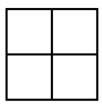
Reinventing the Principal of Mathematical Induction

Our goal is to prove the following claim:

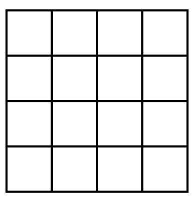
For all $n \in \mathbb{N}$, a $2^n \times 2^n$ grid of squares with exactly one square removed can be tiled using L-shaped tiles of 3 squares.

With your group, explore why this claim is true for $n = 1$, $n = 2$, and $n = 3$. Can you see how the truth of the $n = 2$ case might depend on the $n = 1$ case being true? How might the truth of $n = 3$ depend on $n = 2$ being true?

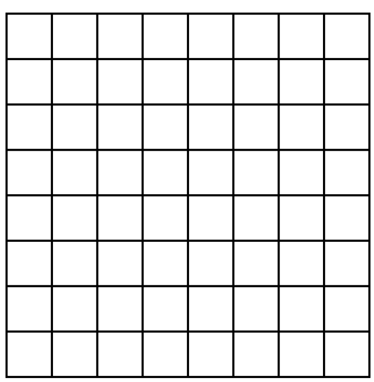
$n = 1$:



$n = 2$:



$n = 3$:



Can you outline the idea behind your group's argument? What are the key logical components that are needed to generalize your argument to prove the claim is true for all $n \in \mathbb{N}$ (not just $n = 1, 2, 3$)?

Investigating the Logical Components of Mathematical Induction

Suppose $P(n)$ is a statement about a positive integer n, and we want to prove:

$$P(n) \textit{ is true for all positive integers } n.$$

Each part below provides given information that is known to be true. For each part, decide with your group whether this information is enough to prove $P(n)$ is true for all positive integers n. If the answer is yes, no justification is necessary. If the answer is no, explain why.

1. $P(1)$ is true; for all integers $k \geq 1$, $P(k)$ is true.
2. $P(1)$ is true; there is an integer $k \geq 1$ such that $P(k) \rightarrow P(k+1)$.
3. $P(1)$ is true; for all integers $k \geq 1$, $P(k) \rightarrow P(k+1)$.
4. For all integers $k \geq 1$, $P(k) \rightarrow P(k+1)$.
5. $P(1)$ is true; for all integers $k \geq 2$, $P(k) \rightarrow P(k+1)$
6. $P(1)$ and $P(2)$ are true; for all integers $k \geq 2$, $[P(k-1) \wedge P(k)] \rightarrow P(k+1)$.

Abstracting the Role of the Inductive Implication

Consider the following argument.

> *Proof.* Let $k \in \mathbb{Z}^+$ be arbitrary. Assume that $k = k + 1$. Then, adding 1 to both sides, $k + 1 = k + 2$. $\square$

What have we proved?

Understanding the Role of k

We have introduced a new variable k in the statement of PMI. Discuss with your group what the role of k is. Why do you think we use k instead of n?

Practicing Proof by Mathematical Induction

1. Prove: *For all* $n \in \mathbb{Z}^{\geq 0}$, 3 *divides* $(2^{2n} - 1)$.

2. Prove: *For all* $n \in \mathbb{Z}^{+}$, $\left(\frac{3}{2}\right)^n \geq 1 + \frac{n}{2}$.

Reinventing Strong Induction

1. Consider the Fibonacci Sequence (f_n) defined as follows.

$$\begin{aligned} f_1 &= f_2 = 1 \\ f_n &= f_{n-1} + f_{n-2} \quad \text{for all } n \geq 3. \end{aligned}$$

Prove: *For all* $n \geq 1$, $f_n < 2^n$.

In the above proof, what were the logical components that we needed to show were true to demonstrate that $P(n)$ was true for all $n \geq 1$?

2. **Sometimes proving the inductive step, i.e. $\boldsymbol{P(k+1)}$ is true, requires a stronger assumption than just "$\boldsymbol{P(k)}$ is true."**

In this case, we must adapt the PMI.

Prove: *For all $n \in \mathbb{Z}^+$ with $n \geq 2$, either n is prime or n is a product of primes.*