Learning Objectives

In this instructional unit, students will

- develop intellectual need for the PMI and learn to reason flexibly with this proof technique.
- play with a geometric task to explore particular cases of a conjecture, focusing on how one case might depend on the previous case.
- analyze the logic of scenarios with given hypotheses (that are adaptations to the hypotheses of the PMI) to determine whether a conjecture must be true for all natural numbers.
- explore and conjecture the necessary logical components (the base case and the universally quantified inductive implication) of the PMI.
- explain the role of the base case and the inductive implication, and select the correct quantification for linking these components together.
- determine whether adaptations to the hypotheses of the PMI are enough to recover the conclusion.
- use correct logical structure together with appropriate mathematical language to write a correct inductive proof.
- recognize and implement appropriate logical adaptations to the PMI to prove conjectures that require stronger assumptions and/or additional base cases.
- articulate the limitations of proof by mathematical induction.

Norms & Heuristics

There will be opportunities to reinforce the following norms and heuristics throughout this lesson:

- Challenging arguments to reach consensus. Students will discuss and determine what the *necessary* logical components are for proving a statement is true for all natural numbers.
- Sharing what you are actually thinking, not just what you think others may want to hear.
- Drawing a picture. Several visualizations of the classic induction metaphors are given throughout the lesson.
- Separating solving for the proof from writing the proof. A lengthy discussion of key ideas is carried out before the instructor writes a complete proof by mathematical induction.

Task Goals & Implementation Notes

Day 1

Task 1. "The L-Tiles Task." Productive engagement with this task should allow **students** to:

- be generative in building the next case from the previous case
- assess whether their argument between specific cases generalizes to work between any two consecutive cases.
- test and adapt their argument as needed when the square that is removed is arbitrarily chosen.
- reflect on the logical structure of quasi-induction.
- provide students with the space to make sense of and summarize the general argument from one case to the next.
- promote student noticing of the role of the base case and how the next case depends on the previous case.
- begin to establish students' intellectual need for the logical components (hypotheses) of PMI.

	Activity	**Timing**
1	Motivating PMI	3 minutes
2	Task 1(a)	Launch: 5 minutes Group work: 15 minutes Whole-class discussion: 15 minutes
3	Task 1(b)	Launch: 3 minutes Group work: 5 minutes Whole-class discussion: 4 minutes

By the end of the whole class discussion, the following **key ideas** should have been addressed:

- A *written* summary of the connection between the $n = 1$ and the $n = 2$ case. Hopefully, splitting the grid into quadrants will be suggested by one of the groups. However, if it is not, you might ask, "did anyone notice that it seems like the $n = 2$ case is just four copies of the $n = 1$ case?" Ask them whether each copy is exactly the $n = 1$ case and what's possibly different.
- It would be tedious to tile the grid by brute force for larger values of n. Thus, proving the claim with the PUG is less than desirable here and motivates searching for connections between cases.
- Showing that 3 divides the number of squares that must be tiled would be insufficient for proving the claim is true. Notice that divisibility does not impose the constraint that the 3-square tile be L-shaped. Ask the class, "Did anyone think about proving this by showing the number of squares that must be tiled is divisible by 3? What did you talk about there?"

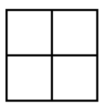
Task 1. Reinventing the Principal of Mathematical Induction

Our goal is to prove the following claim:

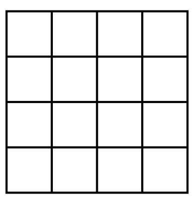
For all $n \in \mathbb{N}$, a $2^n \times 2^n$ grid of squares with exactly one square removed can be tiled using L-shaped tiles of 3 squares.

With your group, explore why this claim is true for $n = 1$, $n = 2$, and $n = 3$. Can you see how the truth of the $n = 2$ case might depend on the $n = 1$ case being true? How might the truth of $n = 3$ depend on $n = 2$ being true?

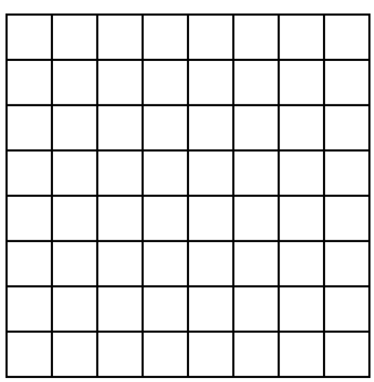
$\underline{\boldsymbol{n = 1}}$:



$\underline{\boldsymbol{n = 2}}$:



$\underline{\boldsymbol{n = 3}}$:



Can you outline the idea behind your group's argument? What are the key logical components that are needed to generalize your argument to prove the claim is true for all $n \in \mathbb{N}$ (not just $n = 1, 2, 3$)?

Day 2

	Activity	Timing
1	Symbolizing and quantifying the inductive implication	Launch: 3 minute Group work: 4 minutes Whole-class discussion: 5 minutes
2	Task 2	Launch: 1 minute Group work: 5 minutes Whole-class discussion: 20 minutes
3	Task 3	Launch: 1 minute Group work: 3 minutes Whole-class discussion: 10 minutes

Task 2. "The Scenarios Task".

Productive engagement with this task should allow students to:

- play with various logical components to determine whether they combine to prove that a statement P (n) is true for all natural numbers n.
- explain why the base case is essential in an inductive argument.
- explain how the quantification of the inductive implication impacts the values of n for which P (n) is true.
- analyze the role of the initial value of k in linking the base case with the inductive implication.
- notice that additional base cases and/or additional hypotheses for the inductive implication might also be used to construct a valid inductive argument. This primes them for strong induction!

By the end of the whole class discussion, the following **key ideas** should have been addressed:

- the staircase metaphor
- the universal quantification of k.
- the values that k should range over, i.e., $1 \leq k < n$.
- For Scenario (1), the language used when instantiating k in the inductive assumption might inadvertently assume the claim itself
- For Scenario (2), existential quantification of $P(k) \to P(k+1)$ is not enough.
- For Scenario (3), these seem like the correct components.
- For Scenario (4), the base case is essential.
- For Scenario (5), the initial value of k is important for linking the base case to the inductive implication.

- For Scenario (6), the base case and/or inductive hypothesis can be adapted to produce another version of PMI.

Task 2:

Suppose $P(n)$ is a statement about a positive integer n, and we want to prove:

$$P(n) \textit{ is true for all positive integers } n.$$

Each part below provides given information that is known to be true. For each part, decide with your group whether this information is enough to prove $P(n)$ is true for all positive integers n. If the answer is yes, no justification is necessary. If the answer is no, explain why.

1. $P(1)$ is true; for all integers $k \geq 1$, $P(k)$ is true.
2. $P(1)$ is true; there is an integer $k \geq 1$ such that $P(k) \rightarrow P(k+1)$.
3. $P(1)$ is true; for all integers $k \geq 1$, $P(k) \rightarrow P(k+1)$.
4. For all integers $k \geq 1$, $P(k) \rightarrow P(k+1)$.
5. $P(1)$ is true; for all integers $k \geq 2$, $P(k) \rightarrow P(k+1)$
6. $P(1)$ and $P(2)$ are true; for all integers $k \geq 2$, $[P(k-1) \wedge P(k)] \rightarrow P(k+1)$.

Task 3. “What Does This Prove?”

Productive engagement with this task should allow students to:

- analyze mathematical language and practice connecting it to mathematical logic.
- abstract the underlying mathematical logic of a written proof.
- determine what mathematical statement is being proved by a given argument.
- Relate the vacuously true implication that was proved to the missing link between the base case and the inductive hypothesis.

By the end of the whole class discussion, the following **key ideas** should have been addressed:

- The language in the proof indicates that an assumption is made and a conclusion is draw from it—this is an implication.
- The logic of the proof is valid, but the assumption is false.
- This is a vacuously true implication because we cannot create a valid counterexample as there is no integer equivalent to the next integer.
- There is a difference between knowing an implication is true and knowing the hypothesis is true.
- The prove does not initialize an inductive algorithm as it does not prove the base case.

Keep prompts very open-ended here and expect a broad variety of ideas to emerge in this discussion.
Ideas for facilitating discussion:

- Invite students to tie their ideas to the staircase analogy. When we tried this, one student said, “we never got on the staircase, so who cares if we can climb from one step to the next.”
- Ask students if they can articulate what seems to be missing from the argument.
- Ask whether the proof follows the structure for proving one of the scenarios in Task 2. The connection to scenario (d) seems to be salient for students.

Task 3:

Consider the following argument.

> *Proof.* Let $k \in \mathbb{Z}^+$ be arbitrary. Assume that $k = k + 1$.
> Then, adding 1 to both sides, $k + 1 = k + 2$. □

What have we proved?

Task 4. “What is k?”

Productive engagement with this task should allow students to:

- notice that n remains fixed throughout the proof, while k varies over values ranging from the base case up to and including $n-1$.
- raise their awareness of the limitations of induction: it cannot prove the infinite case where $n \to \infty$.

The following **key ideas** should be addressed during whole class discussion:

- n is fixed throughout while k captures the variation as you step from the base case to the nth case.
- why denoting the inductive implication as $P(n) \to P(n+1)$ might obscure important mathematical features and limitations of PMI.

Task 4:

We have introduced a new variable k in the statement of PMI. Discuss with your group what the role of k is. Why do you think we use k instead of n?

Day 3

Note: students do not write a formal proof by mathematical induction until Day 3, and only after we have given the statement of the PMI! This write-up is also heavily scaffolded to support attention to the main ideas.

	Activity	**Timing**
1	Students conjecture PMI	Launch: 1 minute Group work: 3 minutes Whole-class discussion: 1 minute
2	State and discuss PMI	5 minutes
3	Task 4	Launch: 1 minute Group work: 1.5 minutes Whole-class discussion: 6 minutes
4	Task 5	Launch: 1 minute Group work: 10 minutes Whole-class discussion: 11 minutes Writing up the proof: 9 minutes

Task 5. Practicing the PMI

Productive engagement with this task should provide students with opportunities to:

- attend to the base case.
- link the base case to the inductive implication with appropriate quantification
- acknowledge that the inductive process concludes with $P(n)$ being true.
- separate solving for the proof from writing the proof.
- engage with the cognitive demand that domain-specific knowledge imposes on proving the inductive implication.
- frame their approach with demonstrating the hypotheses of PMI.

We do not expect students to return to whole class discussion with a perfect, complete proof. The goal of discussion is for students to frame the argument at a high level—not to write a formal proof. The following **key ideas** should be addressed during whole class discussion:

- articulate the two things (in terms specific to this claim) that must be shown to prove the claim is true for all n.
- briefly outline why the base case is true.
- suggest big picture ideas for how the inductive implication might be proven. For example, this may look like a student saying, "we want to use that 3 divides $2^{2k} - 1$ to somehow show that 3 divides $2^{2(k+1)} - 1$.
- state the precise definitions of 3 divides $2^{2k} - 1$ and 3 divides $2^{2(k+1)} - 1$.

Task 5:

Prove that for all $n \in \mathbb{Z}^{\geq 0}$, 3 divides $2^{2n} - 1$.