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Logical implications as mathematical objects: Characterizing epistemological obstacles experienced in introductory proofs courses

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ABSTRACT

Understanding the cognitive challenges students experience in proofs-based mathematics courses is a necessary precursor for supporting them in meeting those challenges. We report on results from a pair of clinical interviews with each of seven STEM majors enrolled in an introductory proofs course. We investigate the epistemological obstacles they experienced in interaction with the interviewer and how those experiences might relate to their treatment of logical implications as actions, objects, or pseudo-objects. We share profiles for each of the seven students, characterizing their treatment of logical implications and their experiences of related epistemological obstacles. These profiles indicate marked differences between epistemological obstacles experienced during interactions with students who treat logical implications as objects, versus actions or pseudo-objects. Results suggest that proof-based mathematics courses should focus centrally on supporting students' constructions of logical implications as mathematical objects.

From a constructivist perspective, engaging all learners requires teachers and researchers to understand students' available ways of operating and to invite students to bring them forth to make meaning in new mathematical contexts (Steffe & D'Ambrosio, 1995). At the same time, teachers and researchers have a role in supporting students' constructions of increasingly powerful ways of operating. In proof-based courses, logical implication plays an especially important role, given the centrality of deductive reasoning in mathematics. A logical implication is a statement of the form "If p is true, then q is true" and is denoted by $p \to q$. Dubinsky (1986) identified two distinct ways that students operate with logical implications: as actions or as objects. As we will demonstrate in this paper, instructors should be sensitive to students who engage in either of these ways of operating and support their construction of the latter.

As an action, a logical implication ($p \to q$) involves three components: a premise, p; a conclusion, q; and a deduction connecting them. Students who treat logical implications as actions can reason by *modus ponens*: the truth of p transfers, by implication, to the truth of q. However, these students might experience persistent challenges in transforming and quantifying logical implications. For instance, they might not reason by *modus tollens*: q being false implies p also must be false ($\sim q \to \sim p$).

As an object, a logical implication is a single entity that a student can act upon. Students who treat logical implications as objects might transform them into equivalent forms, such as the contrapositive, $\sim q \to \sim p$. They might also quantify logical implications as

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wholes, as in “for all x, $p(x) \rightarrow q(x)$.” Importantly, some students might act on logical implications syntactically, without considering their semantic content, in which case we might say they are treating logical implications as pseudo-objects (Zandieh, 2000).

Here, we report on students’ ways of operating with logical implications based on results from case studies with seven STEM majors enrolled in introductory proofs courses. Data come from a pair of clinical interviews with each student, as part of a larger study called the Proofs Project. Within the Proofs Project, we frame epistemological obstacles as persistent challenges experienced in mathematical interactions between teachers and students (Brousseau, 2002).

The purpose of this article is to document epistemological obstacles related to students’ treatment of logical implications as actions, objects, or pseudo-objects. We address the following two research questions:

1. What epistemological obstacles arise during mathematical interactions centered on logical implications and their quantification?
2. How might these epistemological obstacles relate to students’ treatment of logical implications as actions, objects, or pseudo-objects?

We hypothesized we would find differences in the frequency and kinds of epistemological obstacles that arise during interviews with students who treat logical implications as actions, objects, or pseudo-objects. Specifically, we expected to find fewer epistemological obstacles related to transforming and quantifying logical statements, among students who treat logical implications as objects.

1. Theoretical framework

We adopt a Piagetian framework for mathematical development in which students’ logical-mathematical reasoning is understood in terms of their available ways of operating. This overarching framework allows us to integrate the two essential theoretical constructs used in this study: action-object distinctions and epistemological obstacles. Bachelard—the originator of the epistemological obstacles construct—and Piaget corresponded as colleagues, as each pursued a constructivist epistemology: “nothing goes without saying; nothing is given; everything is constructed” (Bachelard, 1934, p. 17). Like Piaget, Bachelard has had an indirect but lasting influence on research in mathematics education, primarily through the work of Sierpińska, Balachef, and Brousseau.

1.1. Action-object theory

Within Piaget’s (1950) epistemology, mathematics is constructed through the coordination of mental actions. Mathematics grows as students’ mental actions—and coordinations thereof—become objects on which to act, opening new possibilities for action and coordination. For example, children construct whole numbers on the basis of their counting activity, which involves forming a one-to-one correspondence between their acts of pointing and their recitation of a verbal number sequence. Ultimately, number words stand in place of acts of pointing so that the child begins to understand 5 as five iterations of an identical unit of 1 (Steffe, 1992). With 5 available as an object, a student can then learn to project five equal parts into a continuous whole, ultimately constructing 1/5 as an object—from a coordination of partitioning a continuous whole into five equal parts and, inversely, reproducing that whole through five iterations of any of those parts.

Piaget’s (1964) epistemology included two senses of objects: (i) they are the products of construction, and (ii) they can be acted upon.

> To know an object, to know an event, is not simply to look at it and make a mental copy or image of it. To know an object is to act on it. To know is to modify, to transform the object, and to understand the process of this transformation, and as a consequence to understand the way the object is constructed. (p. 176)

These two senses of mathematical objects introduce the possibility for a distinction already found in the literature; that of a pseudo-object. The idea of a pseudo-object originates in Linchevsky and Sfard’s (1991) descriptions of “pseudostructural conceptions.” The

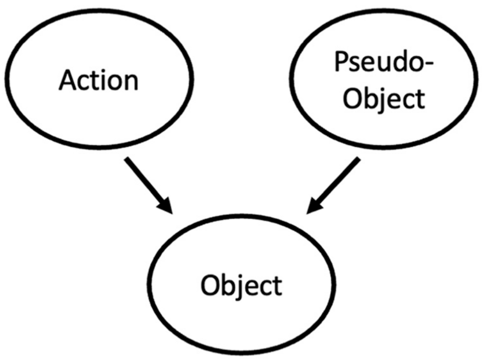

Fig. 1. Relating actions, objects, and pseudo-objects (adapted from Flanagan, 2023).

student "uses symbols as things in themselves and, as a result, remains unaware of the relations between the secondary and primary processes [(ii) and (i), respectively]" (p. 317). Sfard (1992) would later describe these conceptions as "semantically debased" because, although students can act on the symbols as objects, they contain no semantic content. Still later, Zandieh (2000) renamed these semantically debased objects as pseudo-objects, again because they satisfied the second sense of an object (ii) but not the first sense (i). For example, students can become proficient at performing algebraic manipulations on algebraic expressions with little personal meaning for what these expressions might represent (Arcavi, 1994). For these expressions to become mathematical objects in both Piagetian senses, the expressions would serve as proxies for—or pointers to (von Glasersfeld, 1991)—coordinations of actions between objects, as represented by the symbols they contain (Antonides & Battista, 2022; Hackenberg et al., 2021).

Flanagan (2023) characterized the relationships between actions, objects, and pseudo-objects with an illustration similar to the one shown in Fig. 1. An action conception—taken as the coordination of reversible and composable actions that provides the semantic content of a mathematical concept—addresses the first sense of an object. A pseudo-object conception addresses only the second sense. Taken together, these two conceptions form the complete sense of a mathematical object from a Piagetian perspective: a coordination of reversible and composable mental actions, encapsulated, so that a student can act upon it through further action.

1.2. Logical implications as actions or objects

Piaget described the construction of mathematical objects as occurring through a process of reflective abstraction:

Reflective abstraction, which characterizes logico-mathematical thought, draws not from objects but from actions that we can perform on them and, essentially, from the most general coordinations of these actions, such as those of reunion, ordering, and placing in one-to-one correspondence. (Piaget, 1970, p. 18)

Dubinsky (1991) developed APOS theory as an attempt to elaborate on the process of reflective abstraction. In addition to Action and Object phases, he included a Process phase between them and a Schema phase that follows them. In our own use of Piaget's action-object theory, we do not rely upon the Process or Schema phase. However, our action-object framework does align with Dubinsky's broad distinction between students' treatment of logical implications as Actions or Objects, which provided us with the key hypothesis for this study.

Piaget developed action-object theory through his work with children, but Dubinsky's main interest was in applying action-object theory to advanced mathematics. Most of this work focused on proofs-based courses in college mathematics. Dubinsky (1991) hypothesized that for students to develop powerful ways of reasoning about logic and proof (especially with regard to the principle of mathematical induction), they need to treat logical implications as objects, rather than actions. Fig. 2 illustrates the distinction.

A student might treat a logical implication as an action by taking "If p, then q" as a command to act in the following way: determine whether p is true, and if so, conclude that q is true. As such, the implication has three components: two statements (p and q) and an action of inference between them (represented by the arrow between p and q, on the left side of Fig. 2). In contrast, a student who treats a logical implication as an object has encapsulated this way of operating within a single statement (as represented by the rectangle on the right side of Fig. 2). To frame Dubinsky's distinction in the two senses of a mathematical object, we must identify the actions that might comprise a logical implication, as well as the actions one might perform on it.

Drawing on data from their respective research programs, Dawkins and Norton (2022) identified mental actions that students coordinate in making sense of logical implications. We have mentioned the primary action of inference, which relates statement p to statement q. In mathematics courses, we are generally concerned with statements that describe mathematical properties, so inferring refers to content-specific mathematical actions that lead from one property (or set of properties/relationships) to another. For example, in the universe of quadrilaterals, if p is the property of having four congruent sides, it includes the property, q, of having congruent pairs of opposite sides. So, a student can deductively infer, or *deduce*, property q from property p. This action is represented by the rightward arrow between property p and property q in Fig. 3. Inversely (leftward arrow), a student might *abduct* additional properties to specify property p within property q, explaining why property q might occur: this quadrilateral has four congruent sides because it is a square. Note that this logical pattern is reverse that of deduction (Peirce, 1982), so abducting serves well as the inverse action for deducing.

Students can *negate* a property, say p, by switching attention to its complementary property, not p (having four congruent sides versus not having four congruent sides). Students can also use properties to *populate* sets of mathematical objects that satisfy properties p and q (rhombuses and parallelograms); and students can reverse this action by *describing* the properties that members of a set have in common. Within the space of sets (bottom of Fig. 3), *expanding* a set and *nesting* one set within another serve as analogs to *deducing* and *abducting*.

To address the second sense of logical implication as a mathematical object, we consider ways that a student might act on a logical implication as a whole. For example, students might *quantify* a logical implication, as in "for all x in S, $p(x)$ implies $q(x)$." They might *compose* two implications to form a new implication: if $p \rightarrow q$ and $q \rightarrow r$, then $p \rightarrow r$. Finally, they might *transform* a logical implication into a new logical object, such as its converse or its negation. Note that *negating* a logical implication is not the same as negating one or

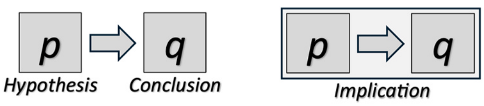

Fig. 2. Logical implication as an action (left) or an object (right).

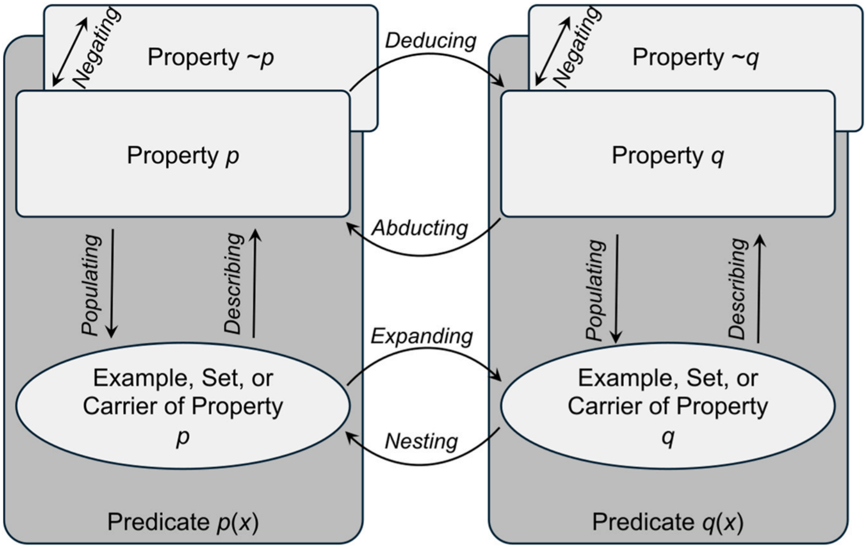

Fig. 3. Actions that comprise logical implications, adapted from Dawkins and Norton (2022).

both of its constituent statements/properties. Rather, it involves producing a statement with the opposite truth value (e.g., the negation of $p \to q$ is p and $\sim q$). We elaborate on these actions, along with research that describes associated challenges, in the next section on epistemological obstacles. Such research also indicates that students can act on logical implications symbolically without the semantic content afforded by the actions that comprise them. This is what we mean when we say a student treats a logical implication as a pseudo-object.

1.3. Epistemological obstacles

The term *obstacle epistemologique* originates in Bachelard's (1938) work on the formation of the scientific mind: "It's in terms of obstacles that we must pose the problem of scientific knowledge" (p. 16). For Bachelard, epistemological obstacles are framed as habits of mind that impede the progress of science, especially as progress occurs through paradigm shifts, such as the transition from Newtonian mechanics to Einstein's theory of relativity. Although Bachelard recognized that existing knowledge forms the basis for new knowledge, he warned that when left unquestioned, existing knowledge can present an obstacle to the development of scientific thought.

Sierpińska (1987) introduced the term "epistemological obstacle" to the mathematics education community, and we find echoes of

Table 1
Epistemological obstacles related to logical implications and their quantification.

Code	Description
Biconditional LI	Interpreting logical implications as bidirectional/biconditional, as in assuming the equivalence of a logical implication ($p \to q$) with its converse, ($q \to p$) or inverse statement ($\sim p \to \sim q$)
Transforming LI	Experiencing difficulty in transforming logical implications into logically related statements, such as their converse, inverse, contrapositive, or negation
Hidden Quantification	Overlooking hidden quantifiers (existential or universal), especially when this leads to ambiguous meaning for a logical statement
Negating Quantification	Demonstrating uncertainty about the quantification of the negation of a statement, especially a logical implication
Negation/Opposite	Treating the negation of an implication, $p \to q$, as its opposite statement, $p \to \sim q$
Multiple Quantification	Struggling to make meaning of statements with two or more quantifiers (i.e., multiply-quantified statements)
Order of Quantification	Questioning how order of quantification might change the meaning of multiply-quantified statements, or assuming that it does not
Role/Value of Variable	Conflating the role of a variable with its value; e.g., attending to the value of a variable, rather than its general role in a statement or proof
PUG	Questioning or neglecting the *principle of universal quantification*, which would allow students to infer that a statement holds *for all* values if it holds *for any* arbitrary value
Euler Representation	Struggling to coordinate logical relationships their spatial relationships in mapping between logical statements and their Euler diagrams
Vacuous Case of LI	Overlooking or ignoring cases in which a logical implication, $p \to q$, is vacuously true because its premise, p, is false
Truth of LI/Premise	Conflating the truth of a logical implication, $p \to q$, with the truth of its premise, p
Quantifying LI/Premise	Quantifying the premise of a logical implication rather than the implication itself
Negation/Disproof	Struggling to discern or articulate the difference between the negation of a statement and showing that the statement is false

Bachelard in Balachef's claim that "old knowledge can turn into an obstacle to the constitution of new conceptions, even though it is a necessary foundation" (1990, p. 264). Brousseau (2002) then distinguished three kinds of obstacles in the construction of new concepts: ontogenetic, didactical, and epistemological. Ontogenetic obstacles refer to knowledge that depends upon further biological maturation, such as brain development. Didactical obstacles refer to shortcomings of instruction in meeting students' pedagogical needs.

In Brousseau's framing, epistemological obstacles are challenges to learning that remain even when biological prerequisites and instructional demands are met. Thus, epistemological obstacles are necessary challenges that persist even in response to research-based instruction (Steffe, 1992 regarding "necessary errors"). They often manifest as tensions between the instructional goals of the teacher and the mathematical experiences of their students. The teacher is likely to experience these tensions first and may need to elicit them among their students so that students might become explicitly aware of these tensions and begin to address them. Many such tensions have been identified in prior research on students' development of logical proof and reasoning.

2. Literature review: epistemological obstacles experienced in introductory proofs courses

Much research has examined students' and teachers' experiences with proof and proving (for review, see Stylianides et al., 2024). In particular, existing research has documented multiple persistent challenges associated with students' reasoning about logical implications and students' quantifications of logical statements in general. In this section, we frame these challenges as epistemological obstacles, and we summarize how they have been discussed in the research literature. Table 1 provides our definition for each epistemological obstacle, and how we coded each of them in data analysis. The table includes ten epistemological obstacles that we extracted from prior research, along with four new codes that emerged during our analysis (discussed further in the Results section).

Prior research has found that students may initially hold biconditional meanings for logical implications (coded "Biconditional LI"; Epp, 2003; Geis & Zwicky, 1971; Matarazzo & Baldassarre, 2010; Rumain et al., 1983; Wason & Johnson-Laird, 1972). That is, in interpreting an if-then statement, students may assume that the converse of the statement is also true. Such interpretations may emerge as products of everyday language. For instance, "you can watch a movie if you finish your homework" implies that watching a movie necessitates that one has finished their homework (Epp, 2003). Although, formally, such interpretations are logically invalid, (Polya (1954); cited in Wason & Johnson-Laird, 1972) argued that they are often plausible: "If it's good, then it's expensive. It's expensive. Therefore (probably) it's good" (p. 43). Drawing on her personal experience as a teacher, Epp (2003) suggests that such presumptions of biconditionality (as well as other obstacles noted here) persist for many students following formal instruction: "I found that students' difficulties were much more profound than I had imagined" (p. 886).

Students may also struggle to transform logical implications into logically equivalent forms, or they may transform them into a logically inequivalent forms (coded "Transforming LI"; Knuth, 2002). For example, students can struggle to conceptualize the logical equivalence between a logical implication and its contrapositive ($\sim q \rightarrow \sim p$) and its disjunctive ($\sim p \vee q$) forms (Antonini, 2004; Goetting, 1995; Hawthorne & Rasmussen, 2015; Hub & Dawkins, 2018; Stylianides et al., 2004). Hub and Dawkins (2018) found that inviting students to reason about logical quartets—a statement and its converse, inverse, and contrapositive—can promote students to abstract contrapositive equivalence. However, they hypothesized that for students to make such an abstraction, they "must *coordinate their connections* across tasks and representations" (p. 99, emphasis original). The persistent effort required to form such coordinations and connections might explain the persistent challenge of Transforming LIs, even among secondary school mathematics teachers who have been teaching for several years (Knuth, 2002).

Quantification poses additional challenges for students, such as when mathematical statements are not *explicitly* quantified but instead include "hidden quantifiers" (coded "Hidden Quantification"). In particular, hidden quantifiers can make it difficult for students to make sense of the negation of a logical implication, proof by contradiction, and multiply-quantified statements (Shipman, 2016). Especially within truth tables, statements of the form $p \rightarrow q$ are often presented with p and q having their own truth values, devoid of quantified variables. Conventionally, the statement $p \rightarrow q$ is understood by experienced readers as meaning $(\forall x)[p(x) \rightarrow q(x)]$, and its negation as $(\exists x)[p(x) \wedge \sim q(x)]$, but such quantified meanings cannot be taken for granted (Shipman, 2016; see also Durand-Guerrier, 2003; Epp, 2003). Indeed, Shipman's (2016) research on hidden quantification was motivated by her own frustrations as a teacher, especially with regard to textbooks that do take those meanings for granted.

Students often struggle to quantify the negation of a logical implication, in particular (coded "Negating Quantification"), and they may conflate the negation of a logical implication with its "opposite" ($p \rightarrow \sim q$; coded "Negation/Opposite"; Arnold et al., 2024). The negation of the logical implication "if p, then q" is frequently stated as "p and not q," which hides quantification (the former being universal, and the latter being existential). Additionally, Epp (2003) argues that colloquial conventions for negating implications may interfere with constructing a viable *mathematical* meaning for negation. For example, one might say, "If I were Ann, I wouldn't do what she did," which someone else might refute by saying, "No, if you were Ann, you would do exactly what she did" (Epp, 2003, p. 889). Logically, the initial statement is of the form $p \rightarrow q$, and its refutation is of the form $p \rightarrow \sim q$, which has a subtly different logical meaning than $(\exists x)[p(x) \wedge \sim q(x)]$. In this context, the negation might be stated, "You were in Ann's position last year, and you did exactly what she did!"

Several studies have documented persistent challenges related to multiply-quantified statements (e.g., Adiredja, 2021; Dawkins & Roh, 2020; Dubinsky & Yiparaki, 2000; Piatek-Jimenez, 2010; Vroom, 2022). Researchers have focused specifically on students' interpretations of statements of the form "for all… there exists…" (AE) and "there exists … for all…" (EA). For example, consider the statements "for all positive real numbers k, there is a natural number M such that $1/k < M$" and "there is a natural number M such that for all positive real numbers k, $1/k < M$" (Piatek-Jimenez, 2010, p. 46). Piatek-Jimenez found that challenges in distinguishing the different meanings across such pairs of statements persisted even at the end an introductory proofs course. We code these persistent challenges as "Multiple Quantification."

Particular challenges can arise as students determine how or whether the order of quantification changes the meaning of the statement (as in comparing two given examples), in which case we code them as "Order of Quantification." Research suggests that students (a) may not be conscious of how they interpret multiply-quantified statements (Dawkins & Roh, 2020), (b) may seem (to an observer) to ignore quantifiers altogether (Dawkins & Roh, 2020), (c) often find it easier to interpret AE statements than EA statements (Dubinsky & Yiparaki, 2000), and (d) often leverage the particular context to make sense of multiply-quantified statements (Dawkins & Roh, 2020; Dubinsky & Yiparaki, 2000).

Additional research suggests college students may struggle with the use of quantified variables in proofs, specifically in distinguishing the *role* of a variable in a proof, versus its *value* in computation (coded "Role/Value of Variable"). This challenge might be expected given students' long histories of computing in mathematics classes, relative to their brief experience with proofs. Dawkins and Roh (2022) share examples from a teaching experiment, which involved students writing divisibility proofs based on definitions like the following: d divides n if and only if there exists an integer k such that $n = kd$. Students tended to make arguments about divisibility based on computations with particular values of k rather than the fact that such an integer exists. Similar challenges arise in more advanced mathematics courses as students take on epsilon-delta proofs for limits (Swinyard & Larsen, 2012).

A common method for proving a universally-quantified statement is to prove that the statement is true for an arbitrary element from the universe of discourse. This method is called the *Principle of Universal Generalization* (abbreviated and coded as "PUG"; see Copi, 1954). Research suggests that students struggle to make sense of and apply the PUG, especially in the context of proving universally quantified logical implications, such as inductive implications (Dawkins et al., 2022; Norton et al., 2023; Selden & Selden, 2013). Indeed, Norton et al. (2023) found that this struggle persisted among students in their introductory proofs courses and contributed to difficulties in mastering the principle of mathematical induction.

Hub and Dawkins (2018) have promoted Euler diagrams as instructional tools to help students address many of the challenges cited here. Although such visual representations can serve as productive tools for students to make sense of logical statements and ascertain their truth values, such diagrams can pose challenges of their own (coded "Euler Representation"; Calvillo et al., 2006; Rizzo & Palmonari, 2005). Antonides et al. (2024) found that students may conceptualize the truth set of the premise of an implication as *containing* the truth set of the conclusion (contrary to the normative representation). This potential conflation between predicates and their truth sets may arise from their reading of $p \rightarrow q$ as "If you have p, then you have q." Additionally, Antonides and colleagues (2024) found that students may assume circular regions of Euler diagrams are necessarily nonempty, potentially exacerbating challenges regarding the vacuous and biconditional cases of a logical implication.

Four additional codes (not suggested by prior literature) emerged during our analysis of clinical interviews. We list those codes at the bottom of Table 1 and discuss them in the Results section.

3. Methods

Our study is situated within the context of the Proofs Project, which expressly focuses on building models of introductory proofs students' logical reasoning and then leveraging these models to design instructional tools and strategies. The project involved fine-grained analysis of teaching and learning within two sections of introductory proofs courses taught at a large land-grant university in the Southeastern United States: one taught in Fall 2022 by the third author, and the other taught in Spring 2023 by the first author.

A1. *If a number is a multiple of 3, then it is a multiple of 6.*

A2. *If a number is a multiple of 6, then it is a multiple of 3.*

A3. *If a number is not a multiple of 6, then it is not a multiple of 3.*

A4. *If a number is not a multiple of 3, then it is not a multiple of 6.*

B1. *If a triangle is not acute, then it is obtuse.*

B2. *If a triangle is obtuse, then it is not acute.*

B3. *If a triangle is not obtuse, then it is acute.*

B4. *If a triangle is acute, then it is not obtuse.*

C1. *If a quadrilateral is a rectangle, then it is a square.*

C2. *If a quadrilateral is a square, then it is a rectangle.*

C3. *If a quadrilateral is not a square, then it is not a rectangle.*

C4. *If a quadrilateral is not a rectangle, then it is not a square.*

Fig. 4. Quartets of logical implications.

3.1. Instruction on logical implication and quantification

As students' treatment of logical implications and quantification is central to this study, we outline the instructional approaches used to introduce them in each class. Both instructors adopted an inquiry-oriented approach, drawing on much of the research cited in this paper. Neither instructor used truth tables. Instead, following Hub and Dawkins (2018), the instructors invited students to investigate the meanings of logical implications and various related forms, such as converse ($q \rightarrow p$), contrapositive ($\sim q \rightarrow \sim p$), and inverse ($\sim p \rightarrow \sim q$). Often, students relied upon Euler diagrams to visualize and compare such statements.

Prior to instruction on logical implication, Author 3 demonstrated how an Euler diagram might be used to depict the truth set P of an open statement $p(x)$. Students then practiced drawing Euler diagrams for pairs of open statements. Next, Author 3 stated the definition of a logical implication and engaged students with a task adapted from Hub and Dawkins's (2018) study. Students were given three "quartets" of logical implications to analyze (see Fig. 4). Each quartet consisted of a logical implication together with its converse, contrapositive, and inverse statements.

Students first determined whether each logical implication was true or false and then visualized each quartet with an Euler diagram. They were asked to analyze the spatial relationship between the truth sets of the premise and conclusion and how this relationship might be related to the validity of each logical implication. Then, students compared statements both within and across quartets to look for logical similarities and differences.

In the remaining six days of instruction on logical implications, students (dis)proved logical equivalences between a logical implication and its converse, contrapositive, and disjunctive representation ($\sim p \vee q$) by comparing Euler diagrams and the sets of counterexamples for each statement. They determined the negation of a logical implication, assessed the vacuous case, analyzed the validity of "proofs" of logical implications (direct, converse, contrapositive, and contradiction), and studied alternative language that might be used to state a logical implication (e.g. "p is sufficient for q").

Author 1 followed a similar approach but began engaging students in formulating proofs starting in the first week. For example, students were asked how they would prove the claim, "All squares are rectangles, but not all rectangles are squares." The first part of the claim can be reframed as the implication, "if a shape (quadrilateral) is a square, then it is a rectangle." Indeed, students began thinking about the statement in that way. During class discussion, Author 1 suggested students could represent the implications by drawing circles to represent the set of squares and the set of rectangles, with the former contained within the latter. Thereafter, students began creating Euler diagrams as personally meaningful tools for investigating logical implications.

During week 2, Author 1 drew upon the quartets task (see Fig. 4), using the first two quartets in much the same way as Author 3. However, because students spent most of class time discussing and presenting proofs, tasks that promoted discussion about how to interpret and represent logical implications—including comparisons to related forms (e.g., converses, contrapositives, and inverses)—were spread across the first five weeks of the semester. For instance, the second class of the fourth week began with a discussion about what $p \rightarrow q$ means, various ways to prove it, and what would happen if p never happened (cases in which the logical implication is vacuously true).

Both instructors used instruction on logical implication as a launching point for instruction on quantification. In particular, issues of quantification arose from classroom discussion on negating logical implications, wherein a universally-quantified logical implication is negated by the existence of a counter-example. To further engage students in discussion about the importance of quantification (including order of quantification), both instructors used tasks like the one shown in Fig. 5, as adapted from prior research (Piatek-Jimenez, 2010, Vroom, 2022). Similar to the way the prior quartets task (see Fig. 4) encouraged students to compare the meanings of various transformations of a logical implication, quartets of quantification tasks encouraged students to compare differences in meaning under various transformations of quantification.

Evaluate the Following Statements

$\forall$ k$\in\mathbb{R}^+$, $\exists$ N$\in\mathbb{N}$ such that N>1/k.
$\forall$ N$\in\mathbb{N}$, $\exists$ k$\in\mathbb{R}^+$ such that N>1/k.
$\exists$ N$\in\mathbb{N}$ such that $\forall$ k$\in\mathbb{R}^+$ N>1/k.
$\exists$ k$\in\mathbb{R}^+$ such that $\forall$ N$\in\mathbb{N}$ N>1/k.

Let a be *a* point on the unit circle, S^1 in $\mathbb{R}^2$.
$\forall$ a$\in S^1$, $\exists$ b$\in\mathbb{R}^2$ such that |a-b|=1.
$\forall$ b$\in\mathbb{R}^2$, $\exists$ a$\in S^1$ such that |a-b|=1.
$\exists$ a$\in S^1$ such that $\forall$ b$\in\mathbb{R}^2$ |a-b|=1.
$\exists$ b$\in\mathbb{R}^2$ such that $\forall$ a$\in S^1$ |a-b|=1.

Fig. 5. Quartets of quantification.

3.2. Data collection

In addition to recordings of whole-class and small-group discussions, project data included video recordings of clinical interviews with individual students (Clement, 2000). Findings from these task-based clinical interviews form the basis for the present study, as reported here. Using recordings from whole-class conversations, we used a technique of purposeful sampling (Maxwell, 2013) to select potential interview participants who would provide diverse ways of logical reasoning and would articulate such reasoning. We found 7 such students who agreed to participate in a sequence of three clinical interviews. Information about each student is provided in Table 2.

Each interview lasted about 45 minutes and was video recorded to capture students' interactions with the interviewer. The first interview focused on students' reasoning with logical implications and occurred during weeks 3–5 each semester. The second interview focused on quantification of variables within logical statements (including logical implications) and occurred during weeks 5–8 each semester. The third interview employed stimulated recall (Calderhead, 1981), using video clips from the first two interviews to question students about their prior responses and underlying reasoning; it occurred at the end of the semester.

We report results from the first two interviews with each student because (1) these interviews used tasks designed to elicit epistemological obstacles identified in prior research (as indicated in Table 3, some of the tasks were borrowed from that research) and (2) these interviews occurred immediately after classroom instruction focused on logical implication and quantification, respectively. As such, the epistemological obstacles students experienced during those interviews might be rightly deemed epistemological (rather than didactical), as they persisted despite our research-based instruction.

Each task for the first two interviews was given separately on slips of paper, and students were provided with a Livescribe smartpen to record what they wrote and drew. The Livescribe smartpen allowed us to capture students' inscriptions as they were created in real time, synced with an audio recording of their speech. Across semesters, tasks differed but maintained the two broad foci within each of the two interviews; within each semester, tasks were the same for each student. Table 3 illustrates sample tasks from each semester/interview. Task names indicate the interview—logical implication (LI) or quantification (Q)—and semester—Fall (F) or Spring (S)—in which the task was used. Note that Q-F3/Q-S5 was used during the quantification interview during both the Fall and Spring.

As indicated in works cited, many of our tasks had been used in prior research. As noted in our review of literature, those studies had used tasks to elicit epistemological obstacles related to logical implications and quantification of logical statements. We used the tasks for the same purpose and designed some of our own tasks to supplement them. For example, we designed Task LI-S1 to assess ways students might transform and compose logical implications.

3.3. Data analysis

Data analysis of the clinical interviews consisted of two parts. First, we coded the epistemological obstacles that each student exhibited during their interviews. Second, we engaged in conceptual analysis to develop epistemic profiles for each student with respect to their treatment of logical implications as actions, objects, or pseudo-objects. We engaged in these two phases of analysis independently to avoid bias in assigning codes to students who treated logical implications as actions, objects, or pseudo-objects.

To code epistemological obstacles, we used the constant comparative method (Corbin and Strauss, 2014), building and iteratively refining a system of codes for categorizing interview data. We began with a draft codebook consisting of the 10 epistemological obstacles drawn from literature reviewed within our theoretical framework. Two researchers (first and second authors) conducted a first round of coding for each student's first two interviews, first analyzing all students' first interviews, then analyzing all students' second interviews. While reviewing the video-recorded interviews, the two researchers built a spreadsheet wherein each line corresponded to specific task or question. Thus, the unit of analysis was an individual's complete response to a particular task or question, and the researchers would not use the same code more than once within such a response. The researchers documented student responses to tasks and questions by recording the corresponding timestamp, any associated codes, a description of the event, and other notes and student quotes, as relevant.

Another pair of researchers (third and fourth authors) engaged in a second round of coding, applying the initial draft codebook to analyze video data student-by-student. This two-stage coding process allowed us to clarify definitions and indicators of codes in the initial draft codebook, define emergent codes as needed, and resolve discrepancies in coding by discussing interpretations of the data with the research team until reaching consensus. In addition to the list of 10 initial codes identified in prior research, 4 new codes emerged from our analysis (see bottom of Table 1). Because new codes emerged from the second round of coding, we did we calculate inter-rater reliability across pairs. Neither did we calculate inter-rater reliability within pairs, because coding occurred collaboratively.

Table 2
Information about Interview Participants.

Pseudonym	Pronouns	Semester	Year	Major
Carmen	She/Her/Hers	Fall 2022	Senior	Engineering
Kai	He/Him/His	Fall 2022	Junior	Data Science
Mary	She/Her/Hers	Fall 2022	Sophomore	Mathematics
Shivani	She/Her/Hers	Fall 2022	Junior	Data Science
Beth	She/Her/Hers	Spring 2023	Junior	Mathematics Education
Erik	He/Him/His	Spring 2023	Senior	Engineering
Zeke	He/Him/His	Spring 2023	Senior	Engineering

Table 3
Sample tasks.

Task	Description
LI-F1	Suppose that the following statement is known to be true. *If two topological spaces are homeomorphic, then their homology groups are isomorphic.* Based on this fact alone, decide whether the following statements are true, false, or uncertain (circle one). a. If two topological spaces have isomorphic homology groups, then the spaces are homeomorphic. True False Uncertain b. If the homology groups of two topological spaces are not all isomorphic, then the spaces are not homeomorphic. True False Uncertain c. There is a pair of homeomorphic topological spaces whose homology groups are not all isomorphic. True False Uncertain (Arnold & Norton, 2017)
LI-F2	Let P and Q be events that have some nonzero probability of occurring, and suppose that the following two implications are true: • If P and Q are mutually exclusive, their probabilities are not independent. • If the probabilities of P and Q are independent, the probability of P and Q is the product of the probability of P and the probability of Q. a. What can you conclude if P and Q are independent? b. What can you conclude if the probability of P and Q is not the product of the probability of P and the probability of Q? c. What can you conclude if P and Q are not mutually exclusive?
Q-F3/ Q-S5	True or false? (Prove or disprove) a. There is a natural number M such that for all positive real numbers k, $1/k < M$. b. For all positive real numbers k, there is a natural number M such that $1/k < M$. (Piatek-Jimenez, 2010)
Q-F5	Part A. Consider the implication $P(x) \rightarrow Q(x)$. What does it mean? How would you quantify it? Part B. Compare the following statements: *For some x*, $P(x) \rightarrow Q(x)$. *For any x*, $P(x) \rightarrow Q(x)$. *For all x*, $P(x) \rightarrow Q(x)$. *For each x*, $P(x) \rightarrow Q(x)$.
LI-S1	Suppose the following statements are true. $P \rightarrow Q$ $Q \rightarrow R$ a. If you know that P is true, what (if anything) can you conclude? b. If you know that Q is true, what (if anything) can you conclude? c. If you know that Q is false, what (if anything) can you conclude? d. If you know that R is true, what (if anything) can you conclude? e. If you know that R is false, what (if anything) can you conclude?
Q-S1	What is the negation of the following statement? If $\lvert x-4 \rvert < 2$, then $-2 < x < 2$.

To assess students' general treatment of logical implications—as actions, objects, or pseudo-objects—we conducted conceptual analyses for the first interview, for each of the seven students. Conceptual analysis is a methodological technique, grounded in constructivist epistemology, that aims to answer the question: "What mental operations must be carried out to see the presented situation in the particular way one is seeing it?" (von Glasersfeld, 1995, p. 78). As explained by Thompson (2008), conceptual analysis enables the constructivist researcher to generate second-order models of students' mathematics; models that can explain how the student might be thinking about a particular idea. Thompson further suggests that conceptual analysis can help researchers explain why students may experience difficulties in understanding particular situations when presented in certain ways. Consistent with these descriptions, we employed conceptual analysis both to understand our students' mathematical worlds and to explain the epistemological obstacles that they seemed to experience.

We operationalized students' treatment of logical implications as actions, objects, or pseudo-objects in the following way. A student with an *action* conception of logical implications can coordinate components of logical implications by acting on them in the ways illustrated in Fig. 3; however, they have not yet encapsulated logical implications as objects on which to act further. So, as evidence for treating logical implications as actions, we looked for indications that students engaged in actions such as deducing one property from another (as in *modus ponens*), negating a property (say, p) to obtain its complementary property ($\sim p$), or populating a set containing elements that share a common property (e.g., the set of all quadrilaterals with four right angles).

Conversely, a student with a *pseudo-object* conception treats logical implications as singular entities on which to act but has not constructed logical implications as coordinations of actions. Evidence of a pseudo-object conception includes acting on representations of logical implications (e.g., symbolic statements or Euler diagrams) by relying on rules for manipulation, seemingly devoid of semantic content. For example, a student with a pseudo-object conception of logical implications might recall that $\sim (p \rightarrow q)$ is equivalent to "$p \wedge \sim q$," but without reasoning through the negation of the implication $p \rightarrow q$.

A student who treats logical implications as objects has constructed them as coordinations of actions *and* conceptualizes them as singular entities on which to act. Evidence of an *object* conception includes transforming and quantifying logical implications as wholes and justifying such holistic actions by referring to the actions that comprise the implication. For instance, a student who treats logical implications as objects could negate the implication $p \rightarrow q$, determining that $\sim (p \rightarrow q)$ is equivalent to "$p \wedge \sim q$," *and* they could justify this relationship by referencing actions on the components p and q. Specifically, they might populate truth sets, P and Q, containing elements that satisfy properties p and q; and then represent the implication $p \rightarrow q$ with an Euler diagram, where the circle representing P is contained within the circle representing Q; noting that this spatial relationship would be negated by the existence of a point inside of P and outside of Q ($p \wedge \sim q$).

4. Results

Here, we report results from the two parts of analysis. The first part—epistemological obstacle coding—enabled us to identify the frequency of various epistemological obstacles experienced during clinical interviews with each student. The second part—conceptual analysis—enabled us to characterize each students' general ways of operating on logical implications, treating them as actions, objects, or pseudo-objects. Bringing these two components of analysis together, we provide profiles (see Table 4) for the epistemological obstacles experienced in interactions with students operating with logical implications as actions, objects, or pseudo-objects.

4.1. Coding of epistemological obstacles

Following the methods described in the Data Analysis section, we produced a spreadsheet of codes, along with timestamps and

Table 4
Frequency of students' experiences of epistemological obstacles.

	Fall 2022				Spring 2023		
Epistemological Obstacle	Carmen	Kai	Mary	Shivani	Beth	Erik	Zeke
Biconditional LI	11 (8,3)	0 (0,0)	6 (6,0)	14 (14,0)	0 (0,0)	2 (1,1)	0 (0,0)
Transforming LI	0 (0,0)	1 (1,0)	5 (4,1)	6 (6,0)	1 (1,0)	5 (5,0)	0 (0,0)
Hidden Quantification	7 (0,7)	3 (0,3)	8 (2,6)	14 (8,6)	1 (0,1)	1 (1,0)	0 (0,0)
Multiple Quantification	0 (0,0)	7 (0,7)	1 (0,1)	2 (0,2)	1 (0,1)	0 (0,0)	0 (0,0)
Quantifying Negations	2 (0,2)	2 (1,1)	1 (0,1)	6 (5,1)	0 (0,0)	2 (1,1)	6 (2,4)
Order of Quantification	4 (0,4)	14 (0,14)	7 (0,3)	10 (0,10)	3 (0,3)	2 (0,2)	0 (0,0)
Role/Value of Variable	4 (0,4)	0 (0,0)	0 (0,0)	4 (0,4)	0 (0,0)	1 (0,1)	0 (0,0)
Negation/Opposite	7 (6,1)	2 (0,2)	8 (0,8)	11 (7,4)	5 (2,3)	4 (3,1)	3 (2,1)
PUG	2 (0,2)	2 (0,2)	2 (0,2)	5 (0,5)	0 (0,0)	0 (0,0)	0 (0,0)
Euler Representation	4 (4,0)	1 (1,0)	6 (5,1)	8 (6,2)	1 (1,0)	3 (1,2)	0 (0,0)
Truth of LI/Premise	1 (0,1)	0 (0,0)	0 (0,0)	0 (0,0)	0 (0,0)	2 (2,0)	0 (0,0)
Vacuous Case of LI	1 (0,1)	3 (2,1)	3 (0,3)	1 (0,1)	7 (4,3)	2 (1,1)	2 (2,0)
Quantifying LI/Premise	0 (0,0)	0 (0,0)	2 (0,2)	0 (0,0)	0 (0,0)	0 (0,0)	0 (0,0)
Negation/Disproof	3 (0,3)	0 (0,0)	1 (0,1)	1 (1,0)	0 (0,0)	1 (1,0)	1 (0,1)

Note: Students who treated logical implications as: actions in white, pseudo-objects in light gray, and objects in dark gray. Numbers in parentheses indicate frequencies from each interview: (LI, Q).

descriptive indicators (key quotes and analytic memos) for each epistemological obstacle. Through our use of the constant comparative method, four additional codes (not suggested by prior literature) emerged during analysis. We hold these challenges as potential epistemological obstacles because we do not yet know whether they will persist in response to research-based instruction intended to address them.

We found that when students considered existentially or universally-quantified logical implications, they sometimes seemed to move the quantification into the implication, applying it to the premise instead (coded Quantifying LI/Premise). This shift in quantification has the effect of reducing the universe of quantification to the truth set of the premise, neglecting cases in which the premise is false, which would render the logical implication vacuously true. Thus, at least from the researcher's perspective, this epistemological obstacle is logically connected to ignoring cases in which a logical implication is vacuously true (coded Vacuous Case of LI). In a recent theoretical paper, Alcock (2025) noted that students might ignore vacuous cases—or even reject vacuously true implications—because the conclusion does not logically follow from the (false) premise. Consequently, a student might negate a universally-quantified logical implication, $(\forall x)[p(x)\rightarrow q(x)]$, by saying $(\exists x)[p(x)\rightarrow \sim q(x)]$ when, in fact, both statements are true when $p(x)$ is false.

Similarly, some students seemed to conflate the truth of a logical implication with the truth of its premise (coded Truth of LI/Premise). Given the truth of $p\rightarrow q$, they would assume the truth of p. In doing so, students would be overlooking cases in which the logical implication is vacuously true because its premise is false. However, this code differs from Vacuous Case of LI code in that students are not simply applying the quantification of the logical implication to its premise; they are applying the truth of the logical implication to its premise.

Finally, we found that some students struggled to differentiate between the negation of a statement (e.g., "there exists an x such that $p(x)$ and not $q(x)$") and the act of showing the statement is false, such as by producing a counterexample to the statement (coded Negation/Disproof). With this final emergent code, we had a complete codebook of 14 epistemological obstacles to apply consistently across all clinical interviews.

With the codebook and spreadsheet complete, we created Table 4 to display the frequency for epistemological obstacles experienced by each student during the clinical interviews. The first number in each cell is the total frequency for the epistemological obstacle, across the two interviews; the numbers in parentheses specify code frequencies within each interview. Here, we provide two examples to demonstrate our coding method and to illustrate ways students experienced epistemological obstacles.

Our analysis of Carmen's responses to tasks during the first interview provided evidence for eight distinct instances in which she seemed to experience challenges that fit the Biconditional LI code. For example, in response to Task LI-F1b, Carmen correctly determined that if the conclusion to the second given logical implication were false (as stated), then its premise would also be false (the probabilities of P and Q are not independent). Thus, we might infer that Carmen had transformed the second given implication into its contrapositive and applied *modus tollens*. However, she then applied her inference that P and Q are not independent to the conclusion of the first given logical implication, determining that its premise (P and Q are mutually exclusive) must be true. In this second determination, Carmen seemed to assume biconditionality of the implication as evidenced by her use of the converse. Furthermore, even her first determination—ostensibly via modus tollens—may have relied on a presumed biconditionality of the second implication. In other words, presumed bidirectionality of the implication equated the truth values of its premise and conclusion, leading Carmen to equate the logical implication with both its contrapositive and its converse.

Our analysis of Kai's responses to tasks during the second interview provided evidence for 21 instances of the Multiple Quantification code (struggling to make sense of multiply-quantified statements). For example, in response to Task Q-F3, Kai determined that the two given statements had "the same meaning but different wording," even though their orders of quantification were reversed. He noticed this reversal but explained that it did not matter, comparing it to the commutativity of conjunctions, like "and," wherein "x and y" means the same thing as "y and x." We note that six of the Multiple Quantification codes assigned to Kai occurred in the context of this task, as he repeatedly struggled to distinguish the meanings of the two statements.

4.2. Conceptual analysis

Relying on our theoretical framework, and following methods described in the Data Analysis section, we conducted a conceptual analysis from the first interview with each student to infer whether they treated logical implications as actions, objects, or pseudo-objects. Shaded columns in Table 4 indicate students who treated logical implications as objects (dark gray), and pseudo-objects (light gray); unshaded columns indicate students who treated logical implications as actions. We share examples to demonstrate each of these classifications, while illustrating the various ways student treated logical implications.

Treating Logical Implications as Actions. Three students—Carmen, Mary, and Shivani—consistently treated logical implications as actions. All three students seemed to treat logical implications as on/off switches wherein if the "if" part of a logical implication switches on, then the "then" part switches on. This way of operating fits an action conception of logical implication: the action of turning on the first part of the logical implication (the premise) is followed by the action of turning on the second part of the logical implication (the conclusion), as linked by an act of deducing. Mary most clearly demonstrated this way of operating in response to Task LI-F2a.

Mary: It's an if statement, so it's either, that happens or it doesn't happen.

Author 1: Oh, I see. So, you are talking about this part happening or not [pointing to the left side of the second logical implication (second bullet point)].

Mary: So, we already know that this part happens because P and Q are independent, and it's asking if they are independent [pointing to the second logical implication]. So, this is true; "if" is a true statement in this case [points thumb up]… "If" happens. But in this one [pointing to the first logical implication], we don't necessarily know if "if" happens because… In my understanding of how if-statements kind of work, being dependent has to depend on being mutually exclusive, but being mutually exclusive does not necessarily mean it's dependent. So, if *P* and *Q* are mutually exclusive… Actually, if they are mutually exclusive… [long pause] No, I'm going to stick with that. If *P* and *Q* are mutually exclusive, they could still be independent… So, we don't know if *P* and *Q* are mutually exclusive, but we do know what their combined probability is, but I can't say anything else because I don't have…

Author 1: Okay, so in addition to knowing that *P* and *Q* are independent, you can also tell me what other fact?

Mary: For "a", the only thing else we know for a fact is that the probability of *P* and *Q* is the product of the probability of *P* and the probability of *Q*.

This way of operating worked well for Mary when the premise was given, as it was for the case of the second bulleted logical implication. However, when the premise was not given, Mary was prone to reversing the logical implication (as she did for the first bulleted logical implication). Also note that she could have applied the truth of statement (a) to the first implication, via its contrapositive, but she did not. We hypothesize that this is because she did not equate the implication with its contrapositive and because the given statement did not match the premise or conclusion in the implication, which would have enabled her to apply her on/off switch.

Mary's reference to logical implication as on/off switches persisted across both interviews and, at times, led her to think of logical implication as biconditional: either the premise and conclusion of a logical implication were both true or both false. With regard to the actions illustrated in Fig. 2, we can say that Mary performed actions of deduction and abduction, between premises and conclusions, often treating them with equal certainty. She treated logical implications themselves as three parts, rather than single wholes: checking the truth value for a premise or conclusion; performing an action of deduction or abduction, and then applying the same truth value of the conclusion or premise. Thus, her way of operating enabled her to engage in *modus ponens* and sometimes *modus tollens*, but at the cost of assuming the converse.

Shivani, too, seemed to treat logical implications as on/off switches, often applying them in reverse order: "In order for *q* to be true, *p* must be true." This way of reasoning fits the action of abduction (Peirce, 1982), explaining that *p* is the reason *q* is true. In the context of Task LI-F1b, she generated the contrapositive by equating the falsity of *p* and *q*.

When prompted, Shivani drew the Euler diagram for $p \rightarrow q$ (see Fig. 6), but she seemed to experience a perturbation (long pauses and expressed uncertainty) when she noticed that the diagram included the possibility that *q* could be true without *p* being true.

Shivani: [Looking at the Euler diagram she just drew, as illustrated in Fig. 6] I think it's confusing me.

Author 1: [Laughs] What is that representing [pointing to Euler diagram]

Shivani:[Laughing] I was saying like 'if *p* is true, then *q* is always true.'

Author 1: Oh, OK!

Shivani: So I put *P* as a sub [circle] of *Q*.

Author 1:So, does that makes sense, that 'if *p*, then *q*' would be represented with *P* as a sub of *Q* [pointing to Euler diagram]

Shivani: [pauses] Um, I think so…but then it doesn't make sense because *q* could be true and not be *p* [pointing to region outside of *P* and inside of *Q*]

This exchange was followed by a long pause during which Shivani stared at the Euler diagram and muttered sub-vocal phrases, like "if *p*, then *q*." Carmen and Mary also produced Euler diagrams, through actions like populating sets and nesting them within each other; and like Shivani, their interpretations of these diagrams were often at odds with their meanings for the logical implications. The use of Euler diagrams was especially problematic for Carmen who sometimes used them to represent properties themselves as nested relations, without ever populating sets associated with each property. For example, in response to Task L1-F1, she represented the implication $p \rightarrow q$ by drawing a circle representing property *q* contained within a circle representing property *p*. Referring to this diagram, she explained that "if I have *p*, then I have *q*" because "*P* engulfs the entire space of *Q*," "but if I don't have *p*, I'm outside of this circle [*P*]… then I definitely do not have *q*." In this way, Carmen used an Euler diagram to equate the validity of *p* with that of *q*.

Finally, in response to Task Q-F5A, all three students seemed to interpret the implication "for all *x*, $P(x) \rightarrow Q(x)$" as "for all *x* in *P*, $Q(x)$." For example, Carmen read the universal quantification as saying, "for every single *x* in the set of *P*." This interpretation can be useful in many cases but overlooks vacuously true cases in which *P(x)* is not true. We understand this interpretation as a consequence of the students' ways of operating. Because these students were expressly focused on whether *P* was true so that they might use the implication to conclude *Q* is true, they did not attend to the vacuous case; for them, in the context of their ways of operating with logical implications as on/off switches, the vacuous case seemed irrelevant.

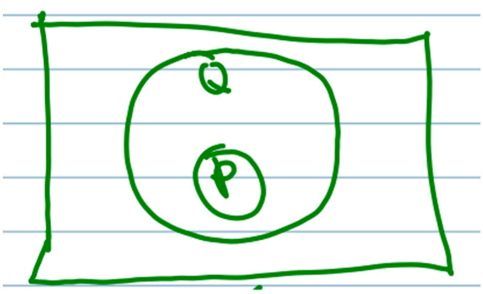

Fig. 6. *Shivani's Euler diagram.*

Treating Logical Implications as Objects. Three students—Beth, Kai, and Zeke—consistently treated logical implications as objects, in both senses of the term. In the first sense, they seemed to have constructed logical implications through a coordination of mental actions and not just symbolic representations to manipulate. Strong evidence for this claim comes from their flexible use of different representations in pointing to the same logical implication. In the second sense, they demonstrated an ability to act on logical implications in various ways, including an ability to compose two implications to form a new implication. Consider the following example of Zeke.

Zeke could represent the same logical implication in multiple ways and could flexibly move back and forth between representations. For example, he saw $p \rightarrow q$ and $\sim q \rightarrow \sim p$ (the contrapositive) as different representations of the same logico-mathematical object: "The contrapositive is just a funny way to say the original statement in terms of 'nots' instead of the actual statement." He could also represent that object spatially by constructing Euler diagrams (set P contained within set Q): "there's something with properties p that would be within Q." The semantic content of the logical implication was comprised of the actions he could perform across and within these various representations, not any particular representation of it. For example, Zeke used *populating* to generate a set P, containing objects, x, that carried the property p: "there's something with property p that would be within Q." In contrast to many students (Dawkins, 2017), Zeke even accepted the *negation* of property p as a property that gave meaning to the truth set for $\sim p$.

Zeke acted on logical implications as objects when he composed, negated, and quantified them. For example, when asked to determine the negation of an implication, $\sim (p \rightarrow q)$, Zeke did not distribute the negation across its components (i.e., $\sim p \rightarrow \sim q$), as found among many students (e.g., Hawthorne & Rasmussen, 2015). Rather, he understood that the implication had to be negated as a whole. Although Zeke (at the time) was unfamiliar with the idea of negating an implication, he generated its meaning during the interview by determining that [p and $\sim q$] would have the opposite truth value of $p \rightarrow q$. Zeke also attended to the quantification of a logical implication as a whole, making distinctions between an existentially-quantified implication and a universally-quantified implication. When given the statement, "If x has property p, then x has property q," and asked, "Must x have property p?", Zeke concluded no: "It's not really saying x has p."

Beth operated similarly to Zeke, constructing multiple representations for the same object (Euler diagrams and contrapositive statements). She used spatial inversion of an Euler diagram to justify that the inverse of an implication is equivalent to its converse; she negated predicates to equate logical implications with their contrapositives; and she composed logical implications to form new logical implications by the "transitive" property. And, like Zeke, she universally quantified logical implications as wholes.

We can see much of this flexibility in representing and acting on logical implications within Beth's response to a single question. When Author 3 asked Beth to generate a counter-example for the implication, "if x has property p. then x has property q," Beth responded, "if it's a counter-example it means x has property p, but it doesn't have property q." She then spontaneously suggested that she could draw a "Venn diagram" (referring to an Euler diagram) to illustrate. She proceeded to draw an Euler diagram, for "if p, then q." like the one Shivani drew (as shown in Fig. 6), but unlike Shivani, Beth experienced no conflict between her logical interpretation of the statement and its spatial representation. Furthermore, she marked an x outside of P and inside of Q to represent her counter-example: "but if there's a counter-example, then there's an x that's in P but not in Q."

Kai, too, operated similarly to Beth and Zeke. He readily mapped logical implications to their spatial representations, engaging in actions of populating and nesting. He flexibly switched back and forth between these representations while making explicit distinctions between logical implications and biconditional statements. Referring to Euler diagrams, he indicated that biconditionality would correspond with the one where sets P and Q were the same circle. Furthermore, Kai used the disjunctive representation a logical implication ($\sim p \vee q$) to reason about its negation ($p \wedge \sim q$).

Treating Logical Implications as Pseudo-Objects. In contrast to the other students, Erik seemed to act on logical implications as objects but did not seem to unpack these objects into constituent actions that would provide semantic content. Thus, we infer that he treated logical implications as *pseudo*-objects. In particular, Erik could compose implications and equate them with their contrapositives: acting on logical implications as wholes. For example, on Task LI-S1a, he reasoned, "Q would have to be true, because P is in Q," and he reasoned that $P \rightarrow Q$ and $Q \rightarrow R$ together mean that $P \rightarrow R$, composing implications.

However, Erik seemed to act on logical implications syntactically (without semantic content) in either of two ways: following rules he had learned relating logical implications to their converses, contrapositives, and inverses; or by mapping to Euler diagrams in a conventional manner and reasoning spatially. In this latter way of operating, Erik did not seem to populate sets. This became problematic when mapping non-standard forms of logical implications to Euler diagrams. For example, on Task LI-S1a, soon after concluding that $P \rightarrow R$, he questioned this conclusion. He seemed to be confused by the Euler diagram of P contained within Q: "Q would have to be true, because P is in Q. But…[pause] this notation is kind of weird, so it might not be that. It might be R [implies] P." He did not draw upon semantic meaning to resolve his confusion, which would require the first sense of a logical implication as an object.

5. Discussion

This study drew upon an epistemological distinction Dubinsky (1986) made between students' treatment of logical implications: as actions or as objects. Drawing further on the underlying Piagetian action-object theory and related work (Flanagan, 2023; Linchevsky & Sfard, 1991; Piaget, 1964, 1970), we have characterized students' treatments of logical implications as actions, objects, or pseudo-objects. These characterizations are based on Piaget's (1964) two senses of object: specifying the actions that constitute a logical implication as an object, and specifying the actions that students might perform on a logical implication as an object. The former sense of object provides logical implications with semantic content, and the latter enables students to transform and quantify them.

This study also drew upon the epistemological work of Bachelard (1934), from whom Sierpińska (1987), Balachef (1990), and

Brousseau (2002) derived the theoretical construct of an epistemological obstacle. In reviewing the literature on persistent challenges students and their instructors experience in introductory courses, we found that many epistemological obstacles pertain to transforming and quantifying statements, especially logical implications. Thus, we formed the theory-based prediction that students who treat logical implications as objects would experience fewer epistemological obstacles associated with proof and proving.

Our results show that ten codes occurred more frequently among students who treat logical implications as actions than among students who treated logical implications as objects: Biconditional LI, Transforming LI, Hidden Quantification, Role/Value of Variable, Negation/Opposite, PUG, Euler Representation, Truth of LI/Premise, Quantifying LI/Premise, and Negation/Disproof (see Table 4). Four of these epistemological obstacles—Biconditional LI, Hidden Quantification, Euler Representation, and Negation/Opposite—occurred more frequently among each of the three students who treated logical implications as actions than they did among any of the three students who treated logical implications as objects (even when combined). With the exception of Carmen's case, this same pattern held for Transforming LI; with the exception of Kai's case, the same pattern held for PUG; and with the exception of Mary's case, the same pattern held for Role/Value of Variable. Note that the probability of this pattern appearing at random is 1 out of 6-choose-3 (5 %), and it held for four of the ten epistemological obstacles found in prior research (Biconditional LI, Hidden Quantification, Negation/Opposite, and Euler Representation) and nearly held for three more epistemological obstacles (Transforming LI, PUG, and Role/Value of Variable). The same pattern might have appeared among three of the four emergent epistemological obstacles (Truth of LI/Premise, Quantifying LI/Premise, and Negation/Disproof) if those codes had occurred more frequently. On the other hand, four codes—Multiple Quantification, Quantifying Negation, Order of Quantification, and Vacuous Case of LI—occurred as frequently among students who treated logical implications as objects as they did among the other students. Thus, we find partial affirmation of the hypothesis that students who treat logical implications as actions are more likely to experience epistemological obstacles associated with transforming and quantifying logical statements.

Table 4 provides an answer to our first research question by indicating the frequency of epistemological obstacles that arose during mathematical interactions centered on logical implications and their quantification. As noted, it also provides support for our hypothesis. We now turn our attention to our second research question, relating students' experiences of epistemological obstacles to their treatment of logical implications as actions, objects, or pseudo-objects.

5.1. Epistemological obstacles associated with treating logical implications as actions

The three students who treated logical implications as actions seemed to share a common way of operating: each of them conceived of a logical implication as a kind of on/off switch. This conception enabled them to reliably apply *modus ponens,* but often led them to equate an implication with its converse (Biconditional LI). In fact, presumed biconditionality occurred uniquely among these students and occurred so often that we might take Biconditional LI as a proxy for an action conception of logical implications.

In treating logical implications as actions, and more specifically as on/off switches, Carmen, Mary, and Shivani were able to apply both *modus ponens* (inferring the truth of the conclusion from the truth of the premise) and *modus tollens* (inferring the falsity of the premise from the falsity of the conclusion, via the contrapositive). However, in so doing, they often assumed the converse (transferring the truth of the conclusion to the truth of the premise) and the inverse (transferring the falsity of the premise to the falsity of the conclusion). We saw Carmen do this in an example shared in the Results section. That example illustrated how students might assume biconditionality of a logical implication (Biconditional LI). We can now understand Carmen's experiences related to the Biconditional LI code as a consequence of her treatment of logical implications as actions, and as on/off switches, in particular.

A second example, from Shivani's first interview, further illustrates how students' experiences of Biconditional LI, Transforming LI, and Euler Representation relate to their treatment of logical implications as actions. Shivani had been considering Task LI-F1. To avoid the unfamiliar context of homology groups, she decided to write the statements in "*P*s and *Q*s": she wrote "if *P*, then *Q*" for the given statement; "if *Q*, then *P*" for statement (a); "$\sim Q$, then $\sim P$" for statement (b); and she wrote "*P*, $\sim Q$" for statement (c), saying "there exists a *P* that's not *Q*." Note that statements (a), (b), and (c) refer to the converse, contrapositive, and negation of the given statement, respectively. Shivani claimed that if the given statement were true, then statements (a) and (b) would have to be true, and that statement (c) would be uncertain. Although she symbolized statements (a) and (b) correctly, Shivani concluded that *both* the converse and the contrapositive were true. That is, Shivani's treatment of logical implications as on/off switches led her to correctly conclude the contrapositive was true, but led her to assume biconditionality (Biconditional LI) when she incorrectly concluded the converse must also be true.

Shivani's responses to Task LI-F1 thus far had provided evidence for her experience of Transforming LI and Biconditional LI. Although she could symbolize the statements correctly, she gave the same truth values to the converse and contrapositive. As with Carmen, and consistent with on/off reasoning, she may have been assuming that the *P*s and *Q*s had to share the same truth values; this gave rise to Biconditional LI. Relatedly, she did not seem to transform the given logical implication as a whole, into its converse and contrapositive, and she did not recognize the negation of the given statement to be false (Transforming LI).

To further probe Shivani's reasoning, the interviewer (Author 1) encouraged her to draw an Euler diagram. Although Shivani drew the correct diagram for the given logical implication (if *P*, then *Q*), she seemed perturbed that it did not also show the converse (if *Q*, then *P*): "I put *P* as a sub[set] of *Q*… [pointing to the diagram] but then it doesn't make sense because *Q* could be true and not be *P*." She spent about five minutes trying to find a suitable Euler diagram that would reflect her symbolic logic before giving up, noting that the two representations were in conflict and that she trusted the symbolic logic more (Euler Representation). Thus, we find a triangle of codes (Biconditional LI, Transforming LI, and Euler Representation) surrounding her treatment of logical implications as actions.

Shivani's example also illustrates a tendency to overlook quantification (Hidden Quantification), particularly among students who treat logical implications as actions. When symbolizing the statements of Task L1-F1, Shivani focused only on labeling the implications.

She did not explicitly denote their quantifications. Although she spoke “there exists” when symbolizing statement (c), she did not engage in comparing the quantification of statements (a)-(c) with the quantification of the given implication. Without quantification, statement (c)’s status as the negation may have been lost, potentially influencing Shivani’s determination that it would be uncertain (rather than false). This provides evidence that students who treat logical implications as actions might not attend to quantification because they are busy coordinating the constituent parts of the logical implications themselves.

We also saw that when students who treated logical implications as actions did attend to quantification, they often quantified logical implications in unconventional ways. In particular, they were prone to moving quantification of a logical implication within the implication, applying it to the premise rather than the logical implication as a whole (Quantifying LI/Premise). Likewise, when negating a logical implication, they tended to move the negation into the implication, applying it to the premise, the conclusion, or both, which sometimes led students to take the opposite of a logical implication, $p \rightarrow \sim q$, as its negation (Negation/Opposite).

Dubinsky et al. (1988) described negation as an action performed on statements, as objects, which might include logical implications. From this theoretical perspective, holding logical implications as objects would be a critical prerequisite development for negating them. The same theoretical explanation could apply to quantification of a logical implication, and indeed, we find that students who treat logical implications as actions experience epistemological obstacles related to negating and quantifying logical implications with greater frequency than students who treat logical implications as objects, particularly Negation/Opposite and Quantifying LI/Premise. Just as these students might negate or quantify the components of a logical implication (instead of the implication itself), they might also transfer the truth of a logical implication to the truth of its components. This could explain students’ experiences related to the Truth of LI/Premise code, wherein they might conflate the truth value of a logical implication with the truth of its premise. In the Results section, our example of Carmen’s response to task Q-F5A illustrated this. These explanations regarding students’ treatment of quantification, negation, and truth values of logical implications—particularly among students who treat logical implications as actions—also resonate with findings from Hawthorne and Rasmussen (2015). Although those authors framed their results differently, they found that such students tend to distribute negations across the components of a logical implication (premise and/or conclusion), leading them to take either the inverse or the opposite of a logical implication as its negation.

5.2. Epistemological obstacles associated with treating logical implications as objects

Holding a logical implication as an object does not preclude all challenges, especially when dealing with quantifiers. Once students begin attending to quantification (addressing Hidden Quantification), they may experience additional challenges, especially Vacuous Case of LI, Multiple Quantification, Quantifying Negation, Order of Quantification. As noted, these four epistemological obstacles were experienced as frequently by students who treated logical implications as objects as they were by students who treated logical implications as actions. Consider, in particular, challenges associated with interpreting statements that have multiple quantifiers: Multiple Quantification and Order of Quantification. Kai experienced persistent challenges associated with both of these codes. For example, consider his response to Task Q-F3/Q-S5.

Recall that we borrowed task Q-F3/Q-S5 from Piatek-Jimenez (2010) who investigated students’ interpretations of “for all… there exists…” (AE) and “there exists … for all…” (EA) statements. Additional research suggests that students find AE statements easier to interpret than EA statements (Dubinsky & Yiparaki, 2000). Kai decided that the first statement (an EA) was true, and justified this response by saying, “however large $1/k$ is, N can be larger.” Thus, it appears he may have interpreted this EA statement as an AE statement, and indeed, he interpreted the second statement (an AE statement) in the same way.

After considering the second statement, Kai determined it was also true because it had “the same meaning but different wording.” When the interviewer (Author 1) asked whether the switch in the order of the quantifiers mattered, Kai responded, “well I would have to think about that for a general case, but I don’t think it matters for here… It’s the exact same thing, so I would say true… I don’t see how the meaning has changed.” Kai had been attentive to the quantification within both tasks and noticed that their order was reversed, but he was inclined to interpret both statements as AE statements (Order of Quantification).

In total, we recorded one Multiple Quantification code and five Order of Quantification codes during Kai’s response to this task, which may be a consequence of his persistence in thinking through and comparing the two statements’ meaning. In fact, later in the interview, while responding to another task, Kai spontaneously returned to thinking about this task, suggesting that the two statements might not have the same meaning after all. Kai’s case suggests that challenges associated with Multiple Quantification and Order of Quantification might persist among students who treat logical implications as objects, because they are willing struggle with multiply-quantified statements and the order of those quantifiers.

Similar conjectures might explain the prevalence of Quantifying Negation and Vacuous Case of LI among students who treat logical implications as objects. Consider, for example, Beth’s response to Task LI-S1. Tasked with finding the negation of the statement, “If $|x - 4| < 2$, then $-2 < x < 2$,” Beth noted that the negation should have opposite truth values from the statement: “so it means ‘not all’, like ‘there exists’… In this case I think it would be ‘there exists an x such that x is not in between -2 and 2’.” Thus, Beth properly quantified the negation but seemed to be taking the truth of the premise ($|x - 4| < 2$) for granted. The interviewer (Author 3) then pressed Beth on this point: “Do I need to say anything about this part” (pointing to the premise). Beth replied, “no… I think with the negation, you’re assuming… Like, if this was ‘If P, then Q,’ then you are assuming P and then showing there’s a possibility for P but not Q.”

Collectively, Beth’s responses provide strong indication that she was not considering cases in which the logical implication was vacuously true (Vacuous Case of LI). Such responses illustrate how a student who treats logical implications as objects can begin to meaningfully transform it into its negation and even successfully navigate its quantification but still overlook more nuanced aspects of logic wherein a logical implication might be true even though its premise is false.

5.3. Epistemological obstacles associated with treating logical implications as pseudo-objects

Finally, we have the lone case of Erik, who treated logical implications as pseudo-objects. Erik seemed to understand logical implications as syntactic objects on which to act, but which lacked the semantic content provided by encapsulating a coordination of actions (i.e., reflective abstraction; Piaget, 1970). Looking back at Table 4, we see that Erik experienced some epistemological obstacles frequently, on par with students who treated logical implications as actions (especially, Transforming LI); he experienced others less frequently, on par with students who treated logical implications as objects (Hidden Quantification, Negation/Opposite, and PUG); and he experienced two more (Biconditional LI and Euler Representation) somewhere in between. In light of our finding that challenges associated with the Biconditional LI code seem to serve as a proxy for treating logical implications as actions, it is especially interesting that Erik experienced such challenges at a frequency between the two groups of students.

Erik did not seem to experience Biconditional LI in the same way that students who treated logical implications as actions did. In particular, he did not treat logical implications as on/off switches as he did not assume that the premise and conclusion of a logical implication should hold the same truth values. Rather, he seemed to follow rules for working with logical implications that were often useful but sometimes in conflict with one another (at least from an observer's perspective). For example, when asked how he might prove a universally-quantified implication, "for all x, $P(x)\rightarrow Q(x)$," Erik responded as follows: "Usually, you don't want to assume $P(x)$ is true, so I'd start saying 'assume like $Q(x)$ is true' or kind of like start with the end of the implication and work backwards to find where P comes out of it." The interviewer (Author 3) thought Erik might be referring to a strategy for generating ideas for a proof by starting from the conclusion, but Erik insisted he would write his proofs by first assuming $Q(x)$: "You could either assume $Q(x)$ is true, and that would lead to possibly a direct proof. Sometimes that's easier to do. And sometimes it's easier to prove that $\sim Q$ is true, and that would lead to the contrapositive." Thus, Erik seemed to be confusing the proof of a logical implication with the proof of its converse, possibly because he was trying to apply a rule for proof by contraposition (starting by assuming Q for a direct proof, just as one might start by assuming $\sim Q$ for a proof of the contrapositive).

The epistemological obstacles Erik experienced were generally the result of applying a rule in an unconventional manner. For instance, he interpreted the Euler diagram for $P(x)\rightarrow Q(x)$ (circle P within circle Q) as saying "an included property of Q is P," rather than saying that the set of objects satisfying property P also satisfy property Q. This explanation appeared to be a consequence of Erik not populating sets with properties P and Q, and Erik used it as further justification for why he would begin his proof that $P(x)\rightarrow Q(x)$ by assuming Q. His explanation is also reminiscent of arguments Carmen made about Q "engulfing" P. As we have noted, at times Carmen also did not populate sets, but she seemed to resolve conflicts between logical representations and Euler diagrams by working though them semantically. In contrast, Erik seemed focused on producing proper notation.

6. Closing

We offer this study as a contribution to the broader work on action-object theory within the math education community. Prior work has demonstrated increased mathematical power as students construct numbers (e.g., Steffe, 1992), algebraic expressions (e.g., Hackenberg et al., 2021), and functions (e.g., Sfard, 1992) as objects. Likewise, our work suggests that students might experience greater logical power with the construction of logical implications as objects. In particular, such a construction would encapsulate the mental actions that give personal meaning for logical implications while providing students with objects that can be further acted upon, especially in quantifying them, negating them, and transforming them into other logical forms (e.g., contrapositive). Thus, our work suggests that math educators should seek ways to support students' constructions of logical implications as objects, and that such constructions can help students address many of the challenges (epistemological obstacles) they might experience in introductory proofs courses.

However, the present study is limited in two key ways. First, although the first and third authors implemented instruction to address cognitive challenges students commonly experience in introductory proofs courses (as identified in prior research), we did not attempt to address those challenges during the clinical interviews. Rather, our interview tasks were only intended to elicit those challenges. Thus, we hold these challenges tentatively as epistemological obstacles that could be meaningfully addressed through more targeted instruction. Second, and similarly, we did not attempt to promote students' constructions of logical implications as objects from their treatment of logical implications as actions, so we cannot determine how intransigent such conceptions might be. Preliminary findings from existing research suggest that performing spatial actions within Euler diagrams can serve as a proxy for coordinating logical relationships and thereby support such development (Antonides et al., 2024). We hope the present study sets the stage for future research that directly addresses both limitations.

CRediT authorship contribution statement

Norton Anderson: Writing – review & editing, Writing – original draft, Supervision, Resources, Project administration, Methodology, Investigation, Funding acquisition, Formal analysis, Data curation, Conceptualization. **Arnold Rachel:** Writing – review & editing, Supervision, Resources, Project administration, Methodology, Investigation, Funding acquisition, Data curation. **Antonides Joseph:** Writing – review & editing, Writing – original draft, Investigation, Formal analysis, Conceptualization. **Kokushkin Vladislav:** Writing – review & editing, Formal analysis, Data curation.

Declaration of Competing Interest

None to report.

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