Banach-Tarski Paradox

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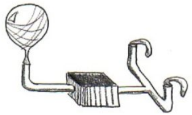
VIRGINIA TECH

Introduction:

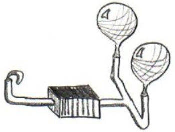
- In 1914, Banach and Tarski proved the Banach-Tarski Paradox [1].
- Using the Axiom of Choice, you can take an object and create two identical copies of the original [1].
- This is proved using 3-D euclidean geometry.

Example:

- This balloon example demonstrates the paradox.



- One balloon with volume v is used to make two balloons with the same volume v.



- This does not work in real life because it is a theoretical math construct [3].

Tools for Key Concept

- Hilbert's Infinite Hotel Principle: If a hotel has an infinite number of rooms with an infinite number of guests then there is always room for another guest. If a guest leaves one of the rooms, then all of the rooms are still filled because there are infinite guests [5].

- Axiom of Choice: If you have a collection infinitely many points, you will always have a point in that collection that is a part of the group of interest. This paradox is built upon the Axiom of Choice, [2].

$$\forall X \left[\varnothing \notin X \implies \exists f\colon X \to \bigcup X \quad \forall A \in X \, (f(A) \in A) \right].$$

Key Concept:

Let's consider some starting point on a sphere and call that point x. We know that the identity property holds true for this point because $I(x) = x$.

Therefore, when we consider other transformations, the point x can be considered I due to the fact that $I(x) = x$.

From this point I we can form a set of points that consists of all of the possible combination of transformations possible made of the combination of Left, Right, Up, and Down transformations..

Ex: $RL = I$ so no point can have an R and an L in it.

We can form a collection of points A that has all of these points as a part of the collection of A.

$A = \{I, R, L, U, D, RU, RD, RR, LU, LD, LL, UR, UU, UD\ldots.\}$

From A we can form 4 different subcollections of A which are determined based off of the last transformation that happens to the transformation.

For example collection B would consist of the following:

$B = \{I, L, LL, LU, LD, LLL, LLU, LLD, LLR\ldots\}$

Let's consider what happens when we transform the subcollection B by R

$B = \{\mathrm{RI}, RL, RLL, RLU, RLD, RLLL, RLLU, RLLD\ldots\}$

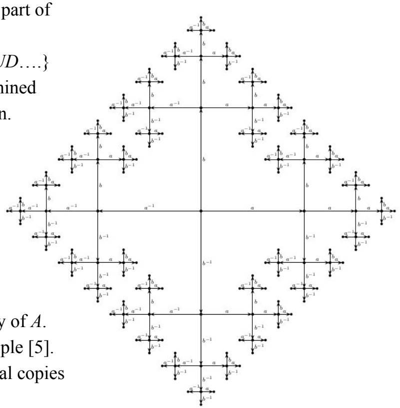
Since R and L are inverse transformations, B then becomes

$RB = \{R, L, U, D, LU, LD, LL, LR\ldots\}$

So now B is both a proper subcollection of A, while also being a copy of A.

This leaves us with a situation just like Hilbert's Infinite Hotel Principle [5].

Using this proof, we can apply it to the sphere and create two identical copies to the first, proving the Banach-Tarski Paradox.



Applications:

- If this were able to be proved using matter, you could create an infinite amount of gold or ice cream!
- Demonstrates how the Axiom of Choice is counterintuitive.[4]
- Shows contradictions in basic geometry.[4]
- Can be used to turn a pea into the size of the sun [2].

Additional Questions:

- Can this be applied in other types of space, such as 4th dimensional euclidean space?
- Does this create a problem with the idea that matter cannot be created or destroyed?
- Every proof deals with a sphere, but can this paradox apply to other 3D objects?

References:

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4. Edward Su, F. (1990). *The Banach Tarski Paradox*. (Minor's Thesis). Harvard University, Cambridge, Massachusetts. Retrieved from https://math.hmc.edu/su/wp-content/uploads/sites/10/2019/06/The-Banach-Tarski-Paradox.pdf.
5. Stevens, M. [Vsauce]. (2015, July 31). *The Banach-Tarski Paradox*. [Video]. YouTube. https://www.youtube.com/watch?v=s86-Z-CbaHA.