Plane Cremona Transformations Fixing Quartic Curves

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Varieties

In algebraic geometry, our fundamental objects of study are *varieties*, the zero sets of polynomials. While we can take polynomials from any ring $R[X_1, X_2, \dots, X_n]$, we often deal with polynomial rings over an algebraically closed field k, such as the complex numbers, and use $\mathbb{P}^n$ as our geometric space. We will begin by stating some fundamental definitions:

- For a polynomial $F \in k[X_1, \dots, X_n]$, we call F a *homogeneous polynomial* if all monomials of F have the same degree. The monomials of F are called *forms*.
- An ideal I is a *homogeneous ideal* if it is generated by a finite set of forms.

A *projective algebraic set* $V(I) \subset \mathbb{P}^n$ is the zero set of a homogeneous ideal. If we cannot separate $V(I)$ into the union of two smaller non-empty algebraic sets, we say that $V(I)$ is *irreducible*. We then define a *projective variety* $V \subset \mathbb{P}^n$ to be an irreducible algebraic set.

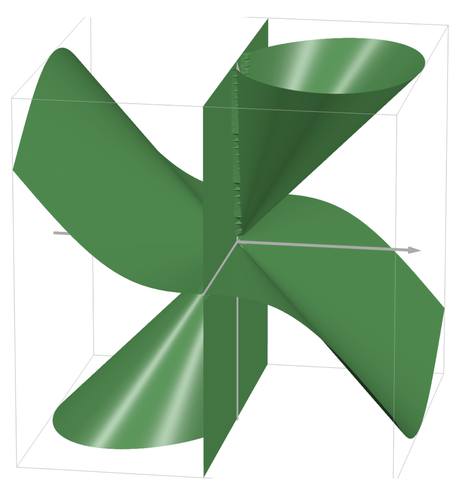

Figure 1. This is the real part of the projective algebraic set $V(X(ZY^2 - Z^2X) + X^4) \subset \mathbb{P}^2$

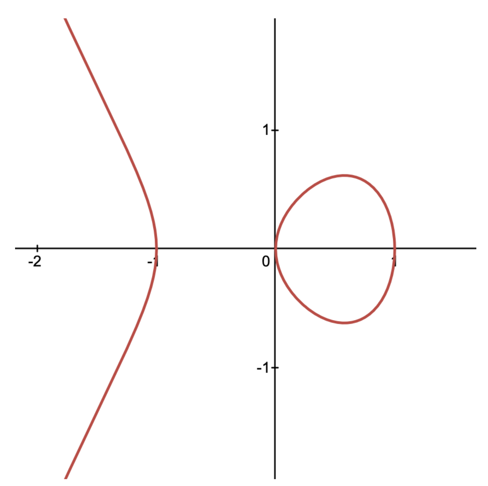

Figure 2. The image of Figure 1 in the open affine plane at Z=1. Note that this algebraic set is **not** a variety

Projective Plane Curves

Remark: When F is a homogeneous polynomial, the ideal generated by F is a homogeneous ideal, $V(I) = V(F)$, and $V(F)$ is a variety.

A *hypersurface* $V(F) \subset \mathbb{P}^2$ is the zero set of a homogeneous polynomial in $k[X, Y, Z]$. This leads to the following formal definition:

A *projective plane curve* is a *hypersurface* in $\mathbb{P}^2$.

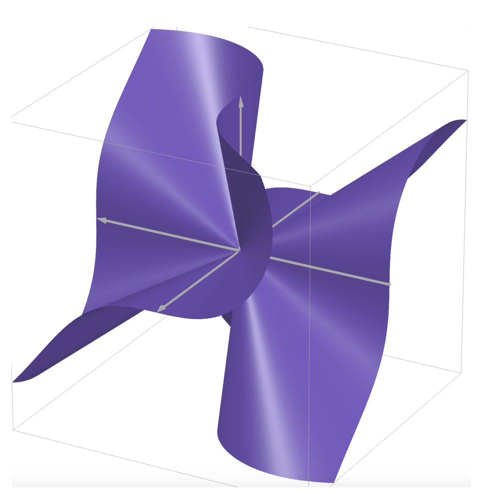

Figure 3. This is the real part of the Klein Quartic projective plane curve $V(X^3Y + ZY^3 + Z^3X)$

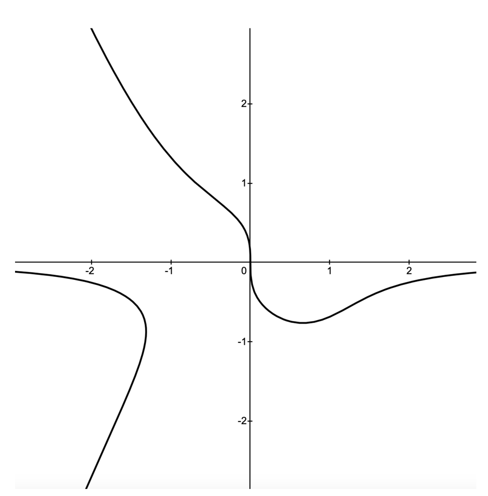

Figure 4. The Klein Quartic curve $V(X^3Y + ZY^3 + Z^3X)$ at Z=1 plotted in the affine plane

Note that the Klein Quartic curve is non singular.

What are Singularities?

A point P of a given plane curve F is called a *multiple point* if

$$F_x(P) = 0 \quad \text{and} \quad F_y(P) = 0,$$

meaning that both partial derivatives of the curve vanish at P. Any curve with such a point is called a *singular curve*, and we call P a *singularity* of F.

We can further classify the type of singularity we have. The multiplicity $m_P(F)$ of a point P on a curve F is the degree of the lowest degree form F_m where

$$F = F_m + F_{m+1} + \cdots + F_n$$

We have that P is a simple point if and only if $m_P(F) = 1$, otherwise P is a double point if $m_P(F) = 2$, a triple point if $m_P(F) = 3$, and so on. A point P is called an ordinary multiple point if $m_P(F) = m$ and there are m distinct tangent lines going through P.

Remark: When F is a projective plane curve, $m_P(F) = m_P(F_*)$ where F_* is the de-homogenization of F.

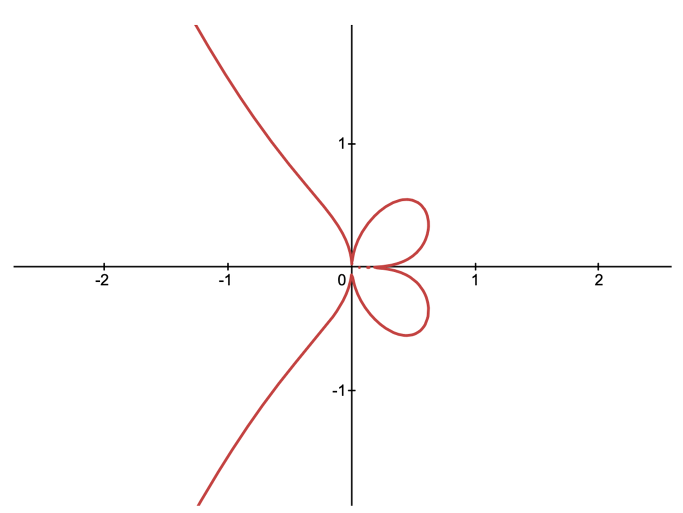

Figure 5. The curve $V((X^3+Y^2)^3 - X^2Y^2)$ plotted in the affine plane. Note that at $P = (0, 0)$ we have a non ordinary quadruple point

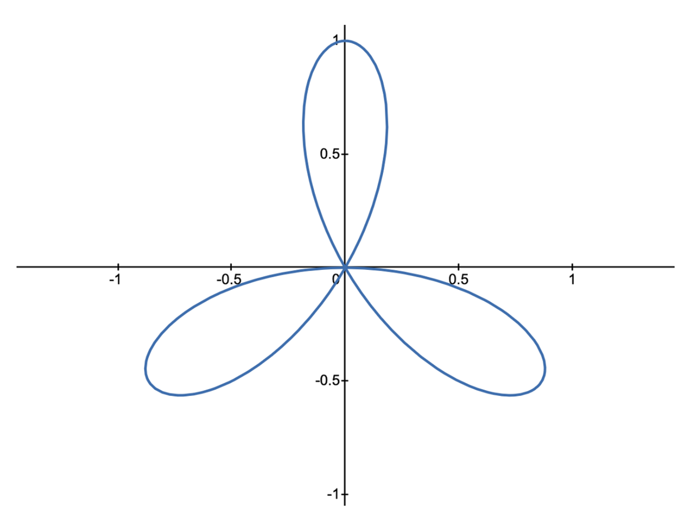

Figure 6. The curve $V((X^2+Y^2)^2 + 3X^2Y - Y^3)$ plotted in the affine plane. Note that at $P = (0, 0)$ we have an ordinary triple point

Mappings and Morphisms

Before we try to resolve a singularity, we must first ask the question, what is a resolution?

First, we define what a birational map is. Let X and Y be varieties. Then a *rational map* $f : X \dashrightarrow Y$ is a morphism from a non-empty open subset $U \subset X$ to Y. A birational map is a rational map f such that f has an inverse rational mapping from $Y \dashrightarrow X$. Note that this inverse mapping need not be defined on all of Y, but only on a non-empty open subset of Y.

We can now define what a resolution of singularities is. To resolve a singularity on a singular projective curve C, we must find a birational map f such that

$$f : C' \to C$$

where C' is a non-singular curve.

Remark: Every variety over a field of characteristic 0 admits a resolution.

Blowing Up Singularities

Blowing up is a transformation used to resolve singularities by replacing a problematic point with an entire geometric object—typically a projective space that captures the limiting directions (tangent directions) at that point.

Formally, blowing up a point on a variety replaces it with the set of directions through that point, creating a new space where the original singularity is spread out and often becomes smooth or simpler to analyze.

How its done

Definition (Affine Blow-up at (0,0) in $\mathbb{C}^2$):

$$\mathrm{Bl}_0(\mathbb{C}^2) = \Big\{((x, y), [u : v]) \in \mathbb{C}^2 \times \mathbb{P}^1 \mid xv = yu\Big\}$$

- $[u : v]$ records the *direction* of the point (x, y).
- The origin is replaced by a full copy of $\mathbb{P}^1$ — the *exceptional divisor*.

Local Charts (Two Affine Patches):

- Chart 1: $u \neq 0 \Rightarrow [1 : t]$, with $y = xt$
- Chart 2: $v \neq 0 \Rightarrow [s : 1]$, with $x = ys$

Each chart gives a new coordinate system where the singularity can be "spread out".

Example

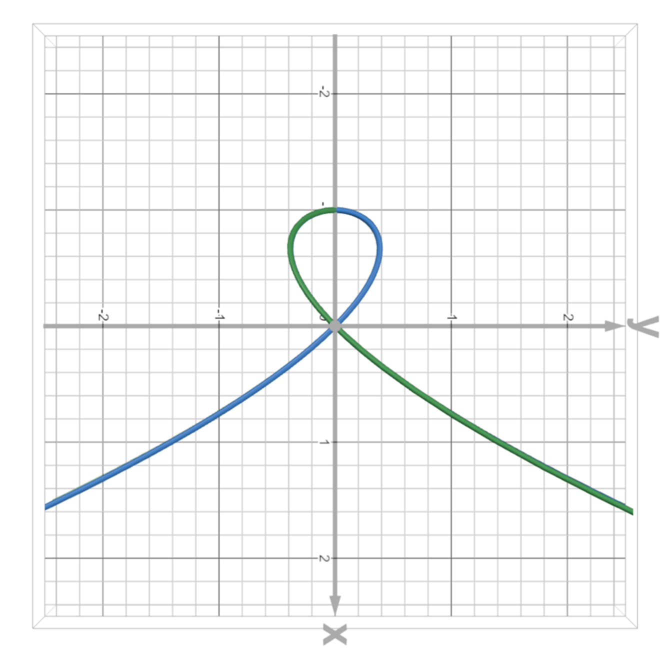

Figure 7. The singular curve $V(X^3 + X^2 - Y^2)$ in the XY-plane

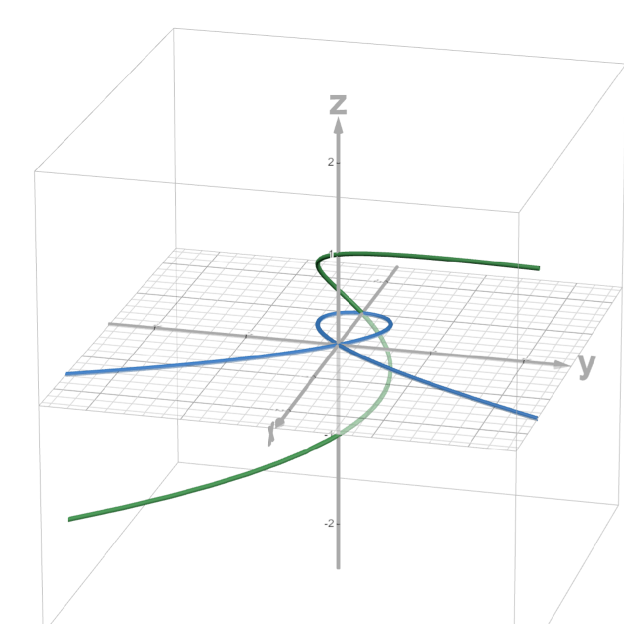

Figure 8. The blown up curve over the singular in the XYZ-space

Blowing up works because it separates the different tangent directions through a singular point. Instead of intersecting at a single problematic point, these directions become distinct points on the new exceptional divisor. In this case the exceptional divisor is $x = 0$ and the two tangent lines at the singularity become two points on the exceptional divisor.

Cremona Transformations

Luigi Cremona

While blowing up helps resolve singularities locally, it doesn't always work globally — especially in complex or higher-dimensional cases. In some situations, singularities persist, or new ones emerge.
This is where Cremona transformations come in. They're not just local fixes, but global birational maps that reconfigure the entire projective plane using a combination of blow-ups and blow-downs. Rather than trying to smooth out a space point-by-point, Cremona transformations can reshape the geometry and turn complicated configurations into simpler ones.

A plane **Cremona transformation** is a birational map $\phi : \mathbb{P}^2 \to \mathbb{P}^2$ given by

$$[x : y : z] \mapsto [f_1(x, y, z), f_2(x, y, z), f_3(x, y, z)]$$

where f_1, f_2, f_3 are all homogeneous polynomials. It is a rational map that is not defined everywhere, but becomes a regular morphism after resolving its indeterminacy via blow-ups.

The Standard Quadratic Cremona Transformation:

$$\phi : [x : y : z] \mapsto [yz : xz : xy]$$

It is a degree two map and it is undefined at the three coordinate points (base points): $[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1]$. Blowing up these points resolves the indeterminacies.

Special Quadratic Cremona Transformations:

While the standard transformation has three proper base points, special quadratic Cremona transformations involve infinitely near points. These points lie on exceptional divisors introduced by earlier blow-ups.

Type I: One Infinitely Near Base Point - One of the base points lies on the exceptional divisor created by blowing up a proper point.

Type II: Two Infinitely Near Base Points - Two base points are infinitely near, often lying on successive blow-ups at the same location.

Problem

Our research focus is to classify plane Cremona transformations fixing a quartic plane curve. More precisely, given a quartic curve C in $\mathbb{P}^2$, we want to classify all plane Cremona transformations Q such that $C \subseteq Q(C)$.

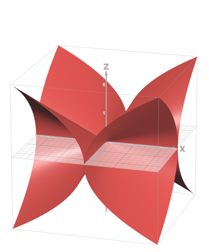

Figure 9. The quartic curve $V(X^2Y^2 - Z^4)$

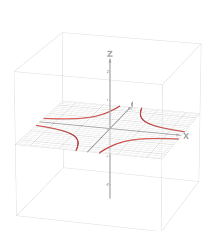

Figure 10. The projection of the quartic curve with $z = 1$

References

[1] Maria Alberich-Carramiñana. *Geometry of the Plane Cremona Maps*. Springer, 1769.
[2] William Fulton. *Algebraic Curves*. 2008.