Monsky's Theorem and Tropical Geometry

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Square Triangulations

- A triangulation of a square is a subdivision of the square into nonoverlapping triangles.

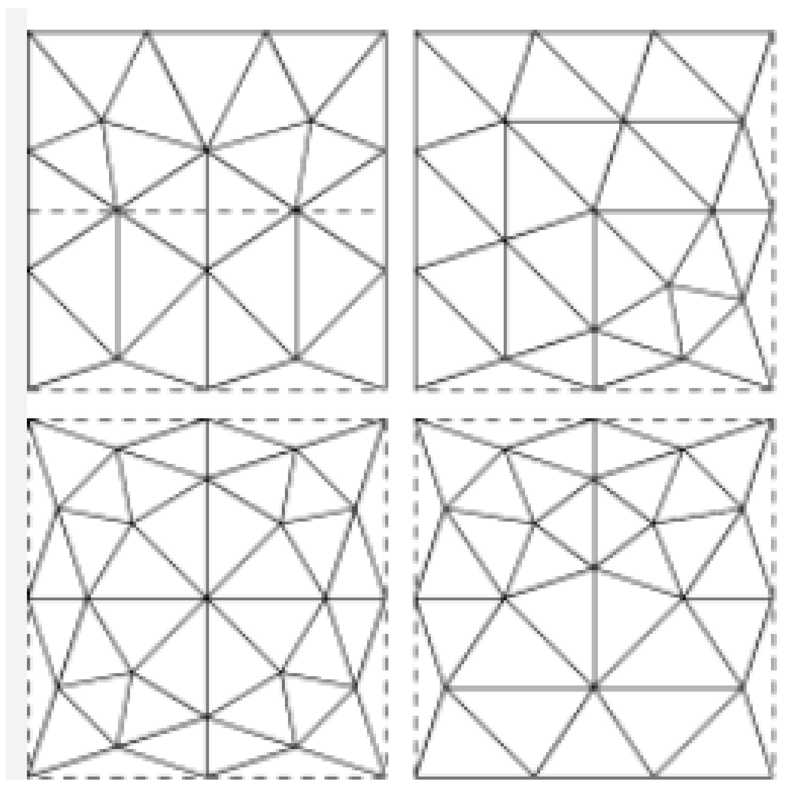

Figure 1:Triangulations of the square

Monsky's Theorem

There exists no triangulation of a square with an odd number of triangles.

Sperner Colorings

- A Sperner coloring on a triangulation of a square is a map from the vertices of the triangulation to 'colors' $\{\mathcal{R}, \mathcal{B}, \mathcal{G}\}$ such that
  1. every edge of the triangulation and the square maps to exactly two colors;
  2. there exists a side of the square σ with endpoints colored $\mathcal{R}$ and $\mathcal{B}$ such that all other sides of the square contain at most one of these colors.
- A triangle given a Sperner coloring is said to be *rainbow* if its vertices are all three colors.

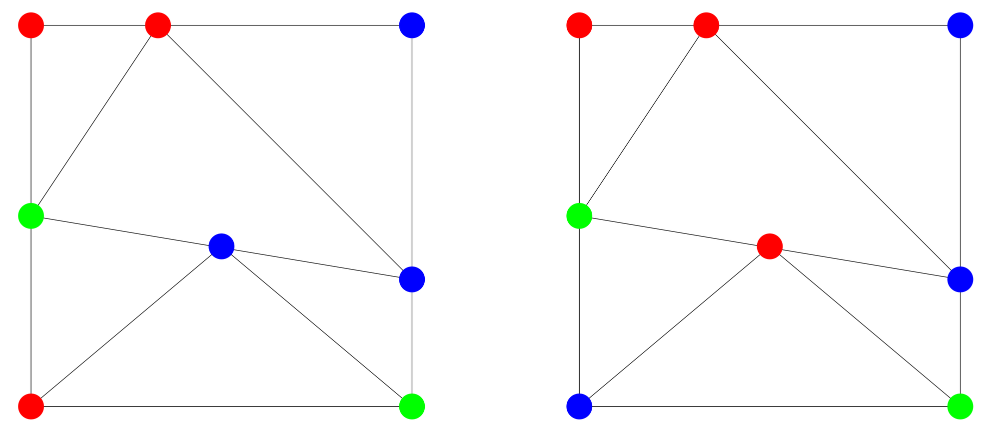

Figure 2:A Sperner coloring (left), and a non-Sperner coloring (right)

Rainbow Triangles

- A 'rainbow triangle' on a Sperner coloring is a triangle with three differently colored vertices. We first must prove its existence by noting that its boundary contains one segment with blue and red vertices (a BR segment), and every other triangle contains an even number of BR segments.
- Observe an odd number of color changes among the vertices on the segment σ alternating between blue and red.
- Every BR segment within the interior is counted twice, for an even sum. By definition of the Sperner coloring, no side besides σ contributes any BR segments, and σ contributes an odd number.
- Thus there exists a rainbow triangle. It remains to show this rainbow triangle cannot have area $\frac{1}{n}$ with n odd.

Tropical Sperner Colorings

- It remains to show that we can produce a Sperner coloring of the vertices of our triangulization.
- We use tropical geometry and first consider Thomas' special case of vertices with rational coordinates.
- Consider the p-adic valuation described in the next column. We assign our vertices without loss of generality to (1,1), (2,1), (1,2), and (2,2) and find the 2-adic valuation on $\mathbb{Q}$ of our vertices of the triangulization.
- Graphing the image of the tropical line $x + y + 1 = 0$ produces a Sperner coloring.
- By inspection, the criteria for the sides are fulfilled. For any arbitrary line in the triangulization, the tropical map produces a point, a parallel line, or a translation of the image, so it intersects at most two of three regions, indicating the edge of a triangle uses only two colors.

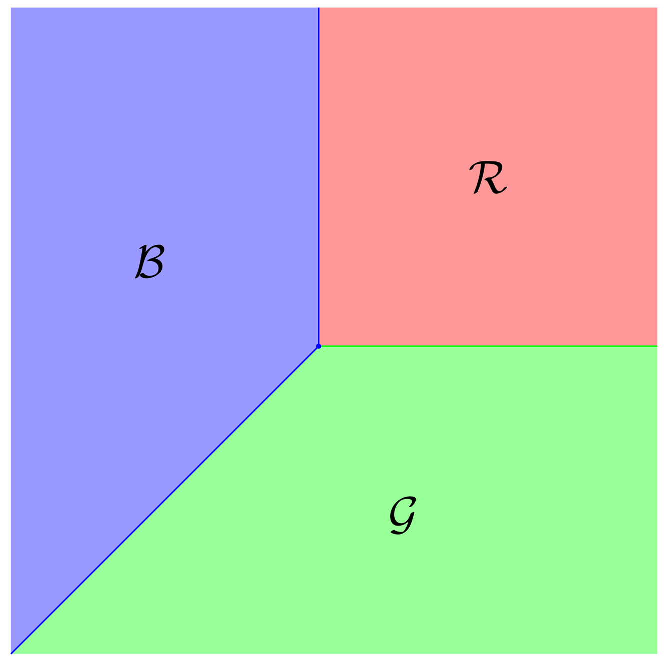

Figure 3:Tropical coloring of the plane divided by $p(x, y) = x \oplus y \oplus 1$.

The Tropical Semiring

- We utilize the tropical semiring $\mathbb{T} = \mathbb{R} \cup \{\infty\}$ with the operations, for $x, y \in \mathbb{T}$,

$$x \oplus y = \min\{x, y\} \quad (1)$$
$$x \otimes y = x + y \quad (2)$$

- The tropical semiring induces a new and interesting geometry to study.

Tropical Plane Curves

A tropical polynomial is a function $p\colon \mathbb{R}^n \to \mathbb{R}$. We consider the case $n = 2$, in which the surface we define from the polynomial to be in the plane. Any such planar polynomial has the form

$$p(x, y) = \bigoplus_{i,j} c_{i,j} \otimes x^i y^j = \min\{c_{i,j} + x^i y^j : i, j\} \quad (3)$$

The surface is defined as the curve on which two monomials of the sum in p reach the same minimum. Consider the following polynomial

$$p(x, y) = x \oplus y \oplus 1 = \min\{x, y, 1\} \quad (4)$$

which graph is the projection of the intersection of the planes x, y, and 1 onto the xy-plane.

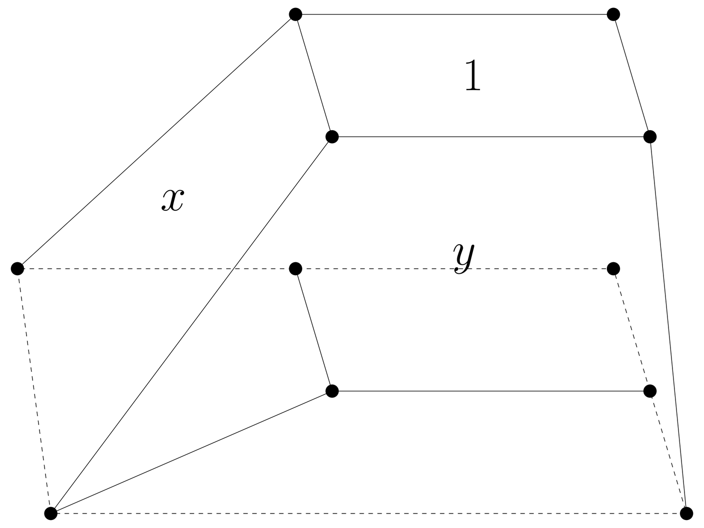

Figure 4:The graph and curve of $p(x, y) = x \oplus y \oplus 1$

Valuations

- Valuations generalize measures of multiplicity.
- For a field K and a totally ordered abelian group Γ, a valuation is a function $\nu\colon K \setminus \{0\} \to \Gamma$ such that
  1. $\nu(ab) = \nu(a) + \nu(b)$
  2. $\nu(a + b) \geq \min\{\nu(a), \nu(b)\}$ with equality if $\nu(a) \neq \nu(b)$
- An example is the well known p-adic valuation $\nu_p\colon \mathbb{Z} \setminus \{0\} \to \mathbb{Z}$ for a prime p. We define, for a nonzero $n \in \mathbb{Z}$,

$$\nu_p(n) = \max\{k \in \mathbb{N} : p^k \mid n\} \quad (5)$$

- The p-adic valuation is a measure of the divisibility of some n by p.

Extending the p-adic Valuation

- For the coloring used to prove Monsky's Theorem, the p-valuation must be extended to the real numbers.
- The valuation can be extended easily to the rationals by defining, for nonzero $a/b \in \mathbb{Q}$,

$$\nu_p(a/b) = \nu_p(a) - \nu_p(b) \quad (6)$$

- It follows from Zorn's Lemma that there exists a transcendental basis T for the field extension $\mathbb{R}/\mathbb{Q}$. This allows us to extend the p-adic valuation to

$$\mathbb{Q}(T) = \operatorname{Frac} \mathbb{Q}[T] = \left\{ \frac{p}{q} : p, q \in \mathbb{Q}[T],\ q \neq 0 \right\} \quad (7)$$

- Each $p \in \mathbb{Q}[T]$ is of the form

$$p = \sum_{j \in J} a_{\vec{e_j}} T^{\vec{e_j}} \quad (8)$$

where each $a_{\vec{e_j}}$ is a coefficient of the monomial $T^{\vec{e_j}}$ for some appropriate exponent vector $\vec{e_j}$ over T. We can define, for nonzero p,

$$\nu_p(p) = \min\{a_{\vec{e_j}} : j \in J\} \quad (9)$$

and, for nonzero $p/q \in \mathbb{Q}(T)$,

$$\nu_p(p/q) = \nu_p(p) - \nu_p(q) \quad (10)$$

- Since $\mathbb{R}/\mathbb{Q}(T)$ is a purely algebraic field extension, there exists a valuation extension to $\mathbb{R}$ from our definitions on $\mathbb{Q}(T)$.

References

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